

CENTRAL LIMIT THEOREM FOR AN ADAPTIVE RANDOMLY REINFORCED URN MODEL

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Abstract. The generalized Pólya urn (GPU) models and their variants have been investigated in several disciplines. However, typical assumptions made with respect to the GPU do not include urn models with diagonal replacement matrix, which arise in several applications, specifically in clinical trials. To facilitate mathematical analyses of models in these applications, we introduce an adaptive randomly reinforced urn model that uses accruing statistical information to adaptively skew the urn proportion toward specific targets. We study several probabilistic aspects that are important in implementing the urn model in practice. Specifically, we establish the law of large numbers and a central limit theorem for the number of sampled balls. To establish these results, we develop new techniques involving last exit times and crossing time analyses of the proportion of balls in the urn. To obtain precise estimates in these techniques, we establish results on the harmonic moments of the total number of balls in the urn. Finally, we describe our main results in the context an application to response-adaptive randomization in clinical trials. Our simulation experiments in this context demonstrate the ease and scope of our model.

1. Introduction. A generalized Pólya urn (GPU) model [4] is characterized by the pair $(Y_{1,n}, Y_{2,n})$ of random variables representing the number of balls of two colors, red and white, for instance. The process is described as follows: at time $n = 0$, the process starts with $(y_{1,0}, y_{2,0})$ balls. A ball is drawn at random. If the color is red, the ball is returned to the urn along with the random numbers $(D_{11,1}, D_{12,1})$ of red and white balls; otherwise, the ball is returned to the urn along with the random numbers $(D_{21,1}, D_{22,1})$ of red and white balls, respectively. Let $Y_{1,1} = y_{1,0} + D_{11,1}$ and $Y_{2,1} = y_{2,0} + D_{12,1}$ denote the urn composition when the sampled ball is red; similarly, let $Y_{1,1} = y_{1,0} + D_{21,1}$ and $Y_{2,1} = y_{2,0} + D_{22,1}$ denote the urn composition when the sampled ball is white. The process is repeated yielding the

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collection $\{(Y_{1,n}, Y_{2,n}); n \geq 1\}$. The quantities $R_1 = \{(D_{11,n}, D_{12,n}); n \geq 1\}$ and $R_2 = \{(D_{21,n}, D_{22,n}); n \geq 1\}$ are collections of independent and identically distributed (i.i.d.) non-negative integer valued random variables, and R_1 is assumed to be independent of R_2 . We refer to

$$D_n = \begin{bmatrix} D_{11,n} & D_{12,n} \\ D_{21,n} & D_{22,n} \end{bmatrix}$$

as a replacement matrix.

In this paper, we focus on an extension of the randomly reinforced urn (RRU) model, a variant of the randomized Pólya urn (RPU) models, whose replacement matrix is given by

$$D_n = \begin{bmatrix} D_{11,n} & D_{12,n} \\ D_{21,n} & D_{22,n} \end{bmatrix} \equiv \begin{bmatrix} D_{1,n} & 0 \\ 0 & D_{2,n} \end{bmatrix}.$$

where the random variables $D_{1,n}$ and $D_{2,n}$ are supported on $[0, \infty)$, rather than on the set of non-negative integers. Let $m_1 := \mathbf{E}[D_{1,n}]$ and $m_2 := \mathbf{E}[D_{2,n}]$. For the RRU model, a law of large numbers was established in [18]; i.e.

$$(1.1) \quad Z_n = \frac{Y_{1,n}}{Y_{1,n} + Y_{2,n}} \xrightarrow{a.s.} \begin{cases} 1 \cdot \mathbf{1}_{\{m_1 > m_2\}} + 0 \cdot \mathbf{1}_{\{m_1 < m_2\}} & \text{if } m_1 \neq m_2, \\ Z_\infty & \text{if } m_1 = m_2, \end{cases}$$

where $\xrightarrow{a.s.}$ stands for almost sure convergence and Z_∞ is a random variable supported on $(0, 1)$. The properties of the distribution of Z_∞ were studied in [1, 2]. Denoting $\{(N_{1,n}, N_{2,n}); n \geq 1\}$ the number of balls of red and white colors sampled from the urn, one can deduce from the above LLN that $N_{1,n}/n$ converges to the same limit as Z_n .

Notice that the limit of the RRU in (1.1) is always 1 or 0 when $m_1 \neq m_2$. However, in applications it is common to target a specific value $\rho \in (0, 1)$. This was achieved in [3], where the modified randomly reinforced urn (MRRU) model was introduced. The MRRU model is an RRU model with two fixed thresholds $0 < \rho_2 \leq \rho_1 < 1$, such that if $Z_n < \rho_2$, no white balls are replaced in urn, while if $Z_n > \rho_1$, no red balls are replaced in the urn. These changes occur at random times and will in general depend on m_1 and m_2 . Thus, even if the sequences $\{D_{1,n}; n \geq 1\}$ and $\{D_{2,n}; n \geq 1\}$ are i.i.d., the replacements matrices of the MRRU model are not i.i.d. Indeed the replacement matrix has the following representation:

$$D_n = \begin{bmatrix} D_{1,n} \cdot \mathbf{1}_{\{Z_{n-1} \leq \rho_1\}} & 0 \\ 0 & D_{2,n} \cdot \mathbf{1}_{\{Z_{n-1} \geq \rho_2\}} \end{bmatrix}.$$

The LLN for the MRRU when $m_1 \neq m_2$ is established as

$$Z_n \xrightarrow{a.s.} \rho_1 \cdot \mathbf{1}_{\{m_1 > m_2\}} + \rho_2 \cdot \mathbf{1}_{\{m_1 < m_2\}}.$$

A second order result for Z_n , namely the asymptotic distribution of Z_n after appropriate centering, was derived in [13]. *However, the validity of the CLT for $N_{1,n}/n$ in the MRRU model is not known.*

A critical issue in the MRRU model is that ρ_1 and ρ_2 are typically unknown in real applications. In this paper, we use the accruing information concerning the balls in the urn to provide random thresholds which converge a.s. to specified targets. More specifically, our replacement matrix becomes

$$(1.2) \quad D_n = \begin{bmatrix} D_{1,n} \cdot \mathbf{1}_{\{Z_n \leq \hat{\rho}_{1,n}\}} & 0 \\ 0 & D_{2,n} \cdot \mathbf{1}_{\{Z_n \geq \hat{\rho}_{2,n}\}} \end{bmatrix},$$

where $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ represent the random thresholds. We call this adaptive urn model an *adaptive randomly reinforced urn* (ARRU), to distinguish it from the RRU and the MRRU. In this paper, we investigate the asymptotic properties of the ARRU model when $m_1 \neq m_2$. Specifically, we establish the LLN for Z_n and $N_{1,n}/n$, and the CLT for $N_{1,n}/n$. Before concluding this section, we describe some recent works in the literature which are similar in spirit to the present work but are quite different from our proposed model.

Let $H_n := \mathbf{E}[D_n | \mathcal{F}_{n-1}]$, where \mathcal{F}_{n-1} is the “information” up to the time $(n-1)$. This is referred to as the generating matrix. Asymptotic properties of the urn composition for homogeneous GPU, i.e. $H_n = H$ for all $n \geq 1$, have been studied in [4] under the assumption that H is irreducible. In [21], the extended Pólya urn (EPU) is defined as a GPU such that all the rows of H sum to the same positive constant, i.e.

$$(1.3) \quad H\mathbf{1} = c\mathbf{1}.$$

Under the assumption that H has simple eigenvalues, second-order asymptotic properties on the proportion of sampled color extracted from the urn are obtained in [21]. In [15], the limiting distribution of the proportion of sampled balls for homogeneous urn models are derived. In [5], weak consistency and asymptotic normality of the urn composition for non-homogeneous GPU are established. However, in [5], the sequence $\{H_n; n \geq 1\}$ is deterministic and converges to a matrix H satisfying (1.3). [6, 7] extend [5] to random generating matrices and establish almost sure convergence of the proportion of sampled balls. They also investigate the second-order properties. A key assumption in [6, 7] is (1.3). In [24] the sequence of generating matrices is defined as function of adaptive estimators, which guarantees the convergence

of H_n to a limiting matrix H satisfying (1.3). For “immigrated” urn models, theoretical results have been obtained in [25] under the assumptions (1.3), or $H\mathbf{1} < 0$. These extensions do not include the RRU model, where H_n is diagonal, non-negative and (1.3) is not satisfied. For distributional results concerning large Pólya urns, see [8]. We now describe application to clinical trial literature (see [11]). For applications to computer science, we refer the reader to [17].

1.1. *Applications to clinical trials.* Urn models have a long history of applications in clinical trials, by providing randomization procedures that target certain objectives (for a review, see [19]). In this context, patients are sequentially allocated to treatments according to the sampled colors and the associated responses are used to update the urn. This is referred to as *response-adaptive*, since the probability of assignment depends on information about the treatment performances. For a literature review on response-adaptive designs in clinical trials see [14, 16]. In an RRU model, responses to treatments are typically transformed by a utility function to obtain the reinforcement values, so that the higher the reinforcement, the better the treatment. This yields a more ethical allocation in clinical trials, because (1.1) shows that the RRU assigns more patients to the superior treatment. However, response-adaptive designs usually aim at obtaining good inferential properties by targeting a certain proportion $\rho \in (0, 1)$, which is typically chosen to satisfy some optimality criteria (see [20]). For this reason, in [3] the RRU was modified to asymptotically attain any target allocation proportion, $\rho \in (0, 1)$. This guarantees the MRRU design to have an asymptotic allocation within $(0, 1)$ there by incorporating ethical constraints (*viz.* assigning more subjects to the superior treatment). The main issue is that ρ_1 and ρ_2 are typically functions of unknown parameters (see [20]). The ARRU model presented in this paper allows ρ_1 and ρ_2 to be functions of such unknown parameters, and adaptively updates by substituting sequential estimates for the parameters. The limiting results in this paper demonstrate that such procedures target the unknown optimal allocation and provide an appropriate randomization procedure for such trials in large samples. We also demonstrate by simulation that the properties hold relatively well for moderate sample sizes.

1.2. *Structure of the paper.* The paper is organized as follows. In Section 2, we present the notation and assumptions concerning the ARRU model and related main results. Specifically, in Subsection 2.1, we present the LLN; in Subsection 2.2 we present the CLT under the assumption that the thresholds are updated at exponentially changing times. Subsection 2.3

is devoted to the implications of the main results in the context of clinical trials.

In Section 3, we describe several fundamental results concerning the ARR model that are needed in the proof of the CLT. Specifically, we prove that the harmonic moments of the total number of balls in the ARR are uniformly bounded. Then, we use this to obtain a uniform L_1 -bound for the distance between the urn proportion at successive update times and the adaptive thresholds. In Section 4 the proofs of the main results are provided, while Section 5 contains results of a simulation study. Section 6 contains extensions to multi-color urn models.

Finally, some remarks concerning proofs are in order. The LLN and CLT for $N_{1,n}/n$ are deduced using the asymptotic properties of Z_n . For this reason, in several results of this paper we will provide a detailed probabilistic description of the sequence $\{Z_n; n \geq 1\}$.

2. Model assumptions, notation and main results. We begin by describing our model precisely. Let $\xi_1 = \{\xi_{1,n}; n \geq 1\}$ and $\xi_2 = \{\xi_{2,n}; n \geq 1\}$ be two sequences of i.i.d. random variables, with probability distributions μ_1 and μ_2 respectively. Without loss of generality (Wlog), assume that the support of $\xi_{1,n}$ and $\xi_{2,n}$ is the same. We denote it by S . Consider an urn containing $y_{1,0} > 0$ red balls and $y_{2,0} > 0$ white balls, and define $y_0 = y_{1,0} + y_{2,0}$. At time $n = 1$, a ball is drawn at random from the urn and its color is observed. Let the random variable X_1 be such that

$$X_1 = \begin{cases} 1 & \text{if the extracted ball is red,} \\ 0 & \text{if the extracted ball is white.} \end{cases}$$

We assume X_1 to be independent of the sequences ξ_1 and ξ_2 . Note that X_1 is a Bernoulli random variable with parameter $z_0 = y_{1,0}/y_0$.

Let $\hat{\rho}_{1,0}$ and $\hat{\rho}_{2,0}$ be two random variables such that $\hat{\rho}_{1,0}, \hat{\rho}_{2,0} \in (0, 1)$ and $\hat{\rho}_{1,0} \geq \hat{\rho}_{2,0}$ a.s. Let $u : S \rightarrow [a, b]$, $0 < a \leq b < \infty$. If $X_1 = 1$ and $z_0 \leq \hat{\rho}_{1,0}$, we return the extracted ball to the urn together with $D_{1,1} = u(\xi_{1,1})$ new red balls. While, if $X_1 = 0$ and $z_0 \geq \hat{\rho}_{2,0}$, we return it to the urn together with $D_{2,1} = u(\xi_{2,1})$ new white balls. If $X_1 = 1$ and $z_0 > \hat{\rho}_{1,0}$, or if $X_1 = 0$ and $z_0 < \hat{\rho}_{2,0}$, the urn composition is not modified. To ease notation, let denote $w_{1,0} = \mathbf{1}_{\{z_0 \leq \hat{\rho}_{1,0}\}}$ and $w_{2,0} = \mathbf{1}_{\{z_0 \geq \hat{\rho}_{2,0}\}}$. Formally, the extracted ball is always replaced in the urn together with

$$X_1 D_{1,1} w_{1,0} + (1 - X_1) D_{2,1} w_{2,0}$$

new balls of the same color to the extracted one; now, the urn composition

becomes

$$\begin{cases} Y_{1,1} = y_{1,0} + X_1 D_{1,1} w_{1,0} \\ Y_{2,1} = y_{2,0} + (1 - X_1) D_{2,1} w_{2,0}. \end{cases}$$

Set $Y_1 = Y_{1,1} + Y_{2,1}$ and $Z_1 = Y_{1,1}/Y_1$. Now, by iterating the above procedure we define $\hat{\rho}_{1,1}$ and $\hat{\rho}_{2,1}$ to be two random variables, measurable with respect to the σ -algebra $\mathcal{F}_1 = \sigma(X_1, X_1 \xi_{1,1} + (1 - X_1) \xi_{2,1})$, with $\hat{\rho}_{1,1}, \hat{\rho}_{2,1} \in (0, 1)$ and $\hat{\rho}_{1,1} \geq \hat{\rho}_{2,1}$ a.s. Let $m_1 = \int u(y) \mu_1(dy)$ and $m_2 = \int u(y) \mu_2(dy)$ be the means of $\{D_{1,n}; n \geq 1\}$ and $\{D_{2,n}; n \geq 1\}$, respectively. We assume throughout the paper the following condition:

$$(2.1) \quad m_1 \neq m_2.$$

The urn process is then repeated for all $n \geq 1$. Let $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ be two random variables, measurable with respect to the σ -algebra

$$\mathcal{F}_n = \sigma(X_1, X_1 \xi_{1,1} + (1 - X_1) \xi_{2,1}, \dots, X_n, X_n \xi_{1,n} + (1 - X_n) \xi_{2,n}),$$

with $\hat{\rho}_{1,n}, \hat{\rho}_{2,n} \in (0, 1)$ and $\hat{\rho}_{1,n} \geq \hat{\rho}_{2,n}$ a.s. We will refer to $\hat{\rho}_{j,n}$ $j = 1, 2$ as threshold parameters. At time $n + 1$, a ball is extracted and let $X_{n+1} = 1$ if the ball is red and $X_{n+1} = 0$ otherwise. Then, the ball is returned to the urn together with

$$X_{n+1} D_{1,n+1} W_{1,n} + (1 - X_{n+1}) D_{2,n+1} W_{2,n}$$

balls of the same color, where $D_{1,n+1} = u(\xi_{1,n+1})$, $D_{2,n+1} = u(\xi_{2,n+1})$, $W_{1,n} = \mathbf{1}_{\{Z_n \leq \hat{\rho}_{1,n}\}}$, $W_{2,n} = \mathbf{1}_{\{Z_n \geq \hat{\rho}_{2,n}\}}$ and $Z_{n+1} = Y_{1,n}/Y_n$ for any $n \geq 1$. Formally,

$$\begin{cases} Y_{1,n+1} = y_{1,0} + \sum_{i=1}^{n+1} X_i D_{1,i} W_{1,i-1} \\ Y_{2,n+1} = y_{2,0} + \sum_{i=1}^{n+1} (1 - X_i) D_{2,i} W_{2,i-1} \end{cases}$$

and $Y_{n+1} = Y_{1,n+1} + Y_{2,n+1}$. If $X_{n+1} = 1$ and $Z_n > \hat{\rho}_{1,n}$, i.e. $W_{1,n} = 0$, or if $X_{n+1} = 0$ and $Z_n < \hat{\rho}_{2,n}$, i.e. $W_{2,n} = 0$, the urn composition does not change at time $n + 1$. Note that condition $\hat{\rho}_{1,n} \geq \hat{\rho}_{2,n}$ a.s., which implies $W_{1,n} + W_{2,n} \geq 1$, ensures that the urn composition can change with positive probability for any $n \geq 1$, since the replacement matrix (1.2) is never a zero matrix. Since, conditionally to the σ -algebra \mathcal{F}_n , X_{n+1} is assumed to be independent of ξ_1, ξ_2 , X_{n+1} is conditionally Bernoulli distributed with parameter Z_n .

We will denote by $N_{1,n}$ and $N_{2,n}$ the number of red and white sampled balls, respectively, after the first n draws, that is $N_{1,n} = \sum_{i=1}^n X_i$ and $N_{2,n} =$

$\sum_{i=1}^n (1 - X_i)$. Let ρ_1 and ρ_2 be two constants such that $0 < \rho_2 \leq \rho_1 < 1$. We will adopt the following notation:

$$\begin{aligned}\hat{\rho}_n &:= \hat{\rho}_{1,n} \mathbf{1}_{\{m_1 > m_2\}} + \hat{\rho}_{2,n} \mathbf{1}_{\{m_1 < m_2\}}; \\ \rho &:= \rho_1 \mathbf{1}_{\{m_1 > m_2\}} + \rho_2 \mathbf{1}_{\{m_1 < m_2\}}.\end{aligned}$$

2.1. *Law of large numbers.* Our first result is concerned with the LLN.

THEOREM 2.1. *Under assumptions (2.1) and*

$$(2.2) \quad \lim_{n \rightarrow \infty} \hat{\rho}_n = \rho \quad a.s.$$

we have that

$$(2.3) \quad \lim_{n \rightarrow \infty} Z_n = \rho \quad a.s.$$

From Theorem 2.1 we can obtain the convergence of sampled balls, namely $N_{1,n}/n$.

COROLLARY 2.1. *Under assumptions (2.1) and (2.2),*

$$(2.4) \quad \lim_{n \rightarrow \infty} \frac{N_{1,n}}{n} = \rho \quad a.s.$$

2.2. *Central limit theorem.* We next study the limit distribution of proportion of sampled balls $\frac{N_{1,n}}{n}$. By the description of the model, $\frac{N_{1,n}}{n}$ depends on the sequence $\hat{\rho}_{j,n}$, $j = 1, 2$. However, frequent changes to $\hat{\rho}_{j,n}$ make the sequence $\frac{N_{1,n}}{n}$ more erratic. To stabilize the behavior of $\left\{ \frac{N_{1,n}}{n}; n \geq 1 \right\}$, we fix a constant $q > 1$ and introduce the sequence $\tilde{\rho}_{j,n}$ $j = 1, 2$ as

$$(2.5) \quad \tilde{\rho}_{j,n} := \hat{\rho}_{j,[q^i]}, \quad \text{as } [q^i] \leq n < [q^{i+1}],$$

for $i \in \mathbb{N}$; that is, we adapt the threshold parameters to change “slowly” at exponential times $\{[q^i], i = 1, 2, \dots\}$. An alternative definition of $\tilde{\rho}_{j,n}$ $j = 1, 2$, which is used in some proofs, is the following

$$(2.6) \quad (\tilde{\rho}_{1,n}, \tilde{\rho}_{2,n}) := \left(\hat{\rho}_{1,[q^{k_n}]}, \hat{\rho}_{2,[q^{k_n}]} \right), \quad k_n := [\log_q(n)],$$

for any $n \geq 1$. We will denote by

$$\tilde{\rho}_n = \tilde{\rho}_{1,n} \mathbf{1}_{\{m_1 > m_2\}} + \tilde{\rho}_{2,n} \mathbf{1}_{\{m_1 < m_2\}}.$$

We now turn to the statement of the CLT. In the following \xrightarrow{d} represents the convergence in distribution.

THEOREM 2.2. *Let $\tilde{\rho}_{1,n}$ and $\tilde{\rho}_{2,n}$ be as in (2.5). Assume that for any $\epsilon > 0$ and $j = 1, 2$, there exists $0 < c_1 < \infty$ such that*

$$(2.7) \quad \mathbf{P}(|\hat{\rho}_{j,n} - \rho_j| > \epsilon) \leq c_1 \exp(-n\epsilon^2),$$

for large n . Then, under assumption (2.1), we have that

$$(2.8) \quad \sqrt{n} \left(\frac{N_{1,n}}{n} - \bar{\rho}_n \right) \xrightarrow{d} \mathcal{N}(0, \rho(1 - \rho)).$$

where $\bar{\rho}_n = \frac{\sum_{i=1}^n \tilde{\rho}_{i-1}}{n}$.

REMARK 2.1. *The result of Theorem 2.2 continues to hold if (2.7) is not satisfied, but (2.2) and the following conditions hold:*

- (c1) $\limsup_{n \rightarrow \infty} \sqrt{n} \mathbf{E}[|\hat{\rho}_n - \rho|] < \infty$.
- (c2) *There exists $\epsilon \in (0, 1/2)$ such that $\hat{\rho}_{j,n} \in [\epsilon, 1 - \epsilon]$ a.s. for any $n \geq 1$, $j = 1, 2$.*

Theorem 2.2 introduces an asymptotic bias for $N_{1,n}/n$ given by $(\bar{\rho}_n - \rho)$. We show that this bias is exactly of order $O(n^{-1/2})$; our next proposition makes this observation precise.

PROPOSITION 2.1. *Let $\tilde{\rho}_{1,n}$ and $\tilde{\rho}_{2,n}$ be as in (2.5). Then, under assumption (2.1) and (2.7),*

$$(2.9) \quad \limsup_{n \rightarrow \infty} n \cdot \mathbf{E}[|\bar{\rho}_n - \rho|^2] < \infty.$$

REMARK 2.2. *The result of Proposition 2.1 continues to hold if (2.7) is not satisfied, but the following condition holds:*

- (c4) $\limsup_{n \rightarrow \infty} n \mathbf{E}[|\hat{\rho}_n - \rho|^2] < \infty$.

In the case when $\hat{\rho}_{1,n} = \rho_1$ and $\hat{\rho}_{2,n} = \rho_2$ for any $n \geq 0$, Theorem 2.2 provides a CLT for the allocation proportion of MRRU model. This is summarized in the following corollary:

COROLLARY 2.2. *In a MRRU, under assumption (2.1), we have that*

$$\sqrt{n} \left(\frac{N_{1,n}}{n} - \rho \right) \xrightarrow{d} \mathcal{N}(0, \rho(1 - \rho)).$$

2.3. *Application to clinical trials (revisited).* Consider two competing treatments \mathcal{T}_1 and \mathcal{T}_2 . The random variables $\xi_{1,n}$ and $\xi_{2,n}$ are interpreted as the potential responses to treatments \mathcal{T}_1 and \mathcal{T}_2 , respectively, given by subjects that sequentially enter the trial. At all times $n \geq 1$, a subject is allocated to a treatment according to the color of the sampled ball and a new response is collected. Note that only one response is observable from every subject, that is $X_n \xi_{1,n} + (1 - X_n) \xi_{2,n}$. The function u transforms the responses into reinforcements $D_{1,n}$ and $D_{2,n}$ that update the urn. Typically, u is chosen such that \mathcal{T}_1 (or \mathcal{T}_2) is considered the superior treatment when $m_1 > m_2$ ($m_1 < m_2$). We assume there exists a unique superior treatment, which is formally stated in assumption (2.1).

We now describe the role of the sequences $\{\hat{\rho}_{1,n}; n \geq 1\}$ and $\{\hat{\rho}_{2,n}; n \geq 1\}$ in clinical trials. Assume the distributions μ_1 and μ_2 are parametric, depending on the vectors θ_1 and θ_2 respectively, with $\theta = (\theta_1, \theta_2) \in \Theta \subset \mathbb{R}^d$, with $d \geq 1$. Let $\hat{\theta}_n = (\hat{\theta}_{1,n}, \hat{\theta}_{2,n})$ be an estimator of θ after the first n allocations, so that $\hat{\theta}_n$ is measurable with respect to the σ -algebra \mathcal{F}_n . We assume that the distributions μ_1 and μ_2 are parametrically independent, in the sense that μ_1 does not depend on θ_2 and μ_2 does not depend on θ_1 . Hence, $\hat{\theta}_{1,n}$ is computed with the $N_{1,n}$ observations $\{\xi_{1,i} : X_i = 1, i \leq n\}$, while $\hat{\theta}_{2,n}$ is computed with the $N_{2,n}$ observations $\{\xi_{2,i} : X_i = 0, i \leq n\}$. Thus, $\{\hat{\rho}_{1,n}; n \geq 1\}$ and $\{\hat{\rho}_{2,n}; n \geq 1\}$ are defined as follows:

$$(2.10) \quad \hat{\rho}_{1,n} := f_1(\hat{\theta}_{1,n}) \quad \text{and} \quad \hat{\rho}_{2,n} := f_2(\hat{\theta}_{2,n}), \quad \forall n \geq 1,$$

where $f_1 : \Theta \rightarrow (0, 1)$ and $f_2 : \Theta \rightarrow (0, 1)$ are two continuous functions such that

$$f_1(x) \geq f_2(x), \quad \forall x \in \Theta;$$

this implies $\hat{\rho}_{1,n} \geq \hat{\rho}_{2,n}$ a.s. for every $n \geq 1$. Moreover, set

$$\rho_1 := f_1(\theta) \quad \text{and} \quad \rho_2 := f_2(\theta).$$

The LLN presented in Theorem 2.1 suggests a direct interpretation for the functions f_1 and f_2 in a clinical trial context: $f_1(\theta)$ and $f_2(\theta)$ represent the desired limiting allocations for the sequence $N_{1,n}/n$, in case the superior treatment is \mathcal{T}_1 ($m_1 > m_2$) or \mathcal{T}_2 ($m_1 < m_2$), respectively. This is a great improvement, since the design can target an arbitrary known function of all the parameters of the response distributions.

Ideally, f_1 and f_2 are chosen to obtain good statistical properties from the design. Typically, in clinical trials, a design is constructed to satisfy certain optimality criteria related to its statistical performances (e.g., power;

see [20]). Letting $\eta(\boldsymbol{\theta})$ denote the limit proportion of subjects to be allocated to treatment \mathcal{T}_1 , this design can be obtained by the urn model described in Section 2 by choosing $f_1(\boldsymbol{\theta}) = f_2(\boldsymbol{\theta}) = \eta(\boldsymbol{\theta})$. However, in some experiments, ethical aspects are important and the main goal may be to assign fewer subjects to the inferior treatment; in this case we choose $f_1(\boldsymbol{\theta}) \simeq 1$ and $f_2(\boldsymbol{\theta}) \simeq 0$. Designs requiring both ethical and statistical goals can also be obtained from our design, by setting $f_1(\boldsymbol{\theta}) \geq \eta(\boldsymbol{\theta}) \geq f_2(\boldsymbol{\theta})$. For instance, we may take

$$(2.11) \quad f_1(\boldsymbol{\theta}) = p \cdot \eta(\boldsymbol{\theta}) + (1-p) \cdot 1, \quad f_2(\boldsymbol{\theta}) = p \cdot \eta(\boldsymbol{\theta}) + (1-p) \cdot 0, \quad p \in (0, 1],$$

where p is a biasing term, which introduces a trade-off between the ethics and statistical properties.

Finally, it is worth emphasizing that conditions (2.2) and (2.7) required in the LLN of Theorem 2.1 and in the CLT of Theorem 2.2, respectively, are straightforwardly satisfied when we take $\hat{\boldsymbol{\theta}}_n$ to be maximum likelihood estimators (MLEs) for $\boldsymbol{\theta}$.

Moreover, condition (c2) in Remark 2.1 is equivalent of the assumption that the ranges of f_1 and f_2 are subsets of $[\epsilon, 1 - \epsilon]$, for some $\epsilon \in (0, 1/2)$.

3. Harmonic moments and related asymptotics.

3.1. Harmonic moments. In this subsection, we show that the harmonic moments of the total number of balls in the urn are uniformly bounded. This is a key result which is needed in several probabilistic estimates, and in particular in the proof of the CLT. More specifically, as explained previously the results concerning the asymptotic behavior of $N_{1,n}/n$, depend critically on the behavior of $(Z_n - \hat{\rho}_n)$. In Subsection 3.2 we provide bounds for $Y_n(Z_n - \hat{\rho}_n)$, by using comparison arguments with the MRRU model. Now, to replace the random scaling Y_n by the deterministic scaling n , one needs to investigate the behavior of n/Y_n . Our next theorem provides a precise estimates of the j^{th} moment of n/Y_n for any $j \geq 0$.

THEOREM 3.1. *Under assumption (2.1) and (2.7), for any $j > 0$, we have that*

$$\sup_{n \rightarrow \infty} \mathbf{E} \left[\left(\frac{n}{Y_n} \right)^j \right] < \infty.$$

In the proof of Theorem 3.1, we need the following lemma that provides an upper bound on the increments of the urn process Z_n , by imposing a condition on the total number of balls in the urn Y_n . Hence, the proof of Theorem 3.1 is reported after the following result.

LEMMA 3.1. *For any $\epsilon \in (0, 1)$, we have that*

$$(3.1) \quad \left\{ Y_n > b \left(\frac{1-\epsilon}{\epsilon} \right) \right\} \subseteq \{ |Z_{n+1} - Z_n| < \epsilon \}.$$

PROOF. The difference $(Z_{n+1} - Z_n)$ can be expressed as follows:

$$\frac{Y_{1,n} + X_{n+1}W_{1,n}D_{1,n+1}}{Y_n + X_{n+1}W_{1,n}D_{1,n+1} + (1 - X_{n+1})W_{2,n}D_{2,n+1}} - \frac{Y_{1,n}}{Y_n}$$

Consider $\{Z_{n+1} > Z_n\}$, since the case $\{Z_{n+1} < Z_n\}$ is analogous. Note that $\{Z_{n+1} > Z_n\}$ implies that $\{X_{n+1} = 1\}$ and $\{W_{1,n} = 1\}$. Then, since $D_{1,n+1} < b$ a.s., on the set $\{Z_{n+1} > Z_n\}$ we have

$$\begin{aligned} Z_{n+1} - Z_n &\leq \frac{Y_{1,n} + D_{1,n+1}}{Y_n + D_{1,n+1}} - \frac{Y_{1,n}}{Y_n} \\ &= \frac{D_{1,n+1}}{D_{1,n+1} + Y_n} (1 - Z_n) \leq \frac{b}{b + Y_n} < \epsilon, \end{aligned}$$

where the last inequality follows from $\{Y_n > b(1 - \epsilon)/\epsilon\}$ in (3.1). \square

PROOF OF THEOREM 3.1. In this proof, when we have set of integers $\{[a_1], \dots, [b_1]\}$ with $a_1, b_1 \notin \mathbb{N}$, to ease notation we will just write $\{a_1, \dots, b_1\}$, omitting the symbol $[\cdot]$. First, note that, since $D_{1,i}, D_{2,i} \geq a$ a.s. for any $i \geq 1$ and $Y_0 > 0$, we have that

$$\begin{aligned} (3.2) \quad Y_n &= Y_0 + \sum_{i=1}^n (D_{1,i}X_iW_{1,i-1} + D_{2,i}(1 - X_i)W_{2,i-1}) \\ &\geq Y_0 + a \cdot \sum_{i=1}^n (X_iW_{1,i-1} + (1 - X_i)W_{2,i-1}), \\ &\geq Y_0 + a \cdot \sum_{i=n\beta}^n (X_iW_{1,i-1} + (1 - X_i)W_{2,i-1}), \end{aligned}$$

for any $\beta \in (0, 1)$. To keep calculation transparent we choose $\beta = 1/2$. We recall that, by construction, we have that $W_{1,i-1}, W_{2,i-1} \in \{0, 1\}$ and $W_{1,i-1} + W_{2,i-1} \geq 1$ for any $i \geq 1$; hence, the random variables $X_iW_{1,i-1} + (1 - X_i)W_{2,i-1}$ are, conditionally to the σ -algebra \mathcal{F}_{i-1} , Bernoulli distributed with parameter greater than or equal to $\min\{Z_{i-1}, 1 - Z_{i-1}\}$. Hence, the behavior of Y_n is intrinsically related to the behavior of Z_n .

Thus, let us introduce the sets $A_{d,n}$ (down), $A_{c,n}$ (center) and $A_{u,n}$ (up) as follows:

$$\begin{aligned} A_{d,n} &:= \left\{ \bigcup_{n/2 \leq i \leq n} \{Z_i < c\} \right\}, \\ A_{c,n} &:= \left\{ \bigcap_{n/2 \leq i \leq n} \{Z_i \in [c, 1-c]\} \right\}, \\ A_{u,n} &:= \left\{ \bigcup_{n/2 \leq i \leq n} \{Z_i > 1-c\} \right\}, \end{aligned}$$

where $c \in (0, 1)$ will be appropriately fixed more ahead in the proof. Then, we perform the following decomposition on the behavior of $\{Z_i; n/2 \leq i \leq n\}$,

$$\mathbf{E} \left[\left(\frac{n}{Y_n} \right)^j \right] \leq \left(\frac{n}{Y_0} \right)^j \cdot \mathbf{P}(A_{d,n}) + \mathbf{E} \left[\left(\frac{n}{Y_n} \right)^j \mathbf{1}_{A_{c,n}} \right] + \left(\frac{n}{Y_0} \right)^j \cdot \mathbf{P}(A_{u,n}).$$

On the set $A_{c,n}$ the process $\{Z_i; n/2 \leq i \leq n\}$ is bounded away from the extreme values $\{0; 1\}$; Hence we can use comparison arguments with a sequence of i.i.d. Bernoulli random variables with parameter c to get the boundedness of $\mathbf{E} \left[(n/Y_n)^j \mathbf{1}_{A_{c,n}} \right]$. After that, we will focus on proving that $\mathbf{P}(A_{d,n})$ and $\mathbf{P}(A_{u,n})$ converge to zero exponentially fast.

First, note that on the set $A_{c,n}$ the random variables

$$X_i W_{1,i-1} + (1 - X_i) W_{2,i-1}$$

are, conditionally to the σ -algebra \mathcal{F}_{i-1} , Bernoulli with parameter with parameter greater than or equal to c for any $i = n/2, \dots, n$. Hence, if we introduce $\{B_i; i \geq 1\}$ a sequence of i.i.d. Bernoulli random variable with parameter c , from (3.2) we have that

$$\mathbf{E} \left[\left(\frac{n}{Y_n} \right)^j \mathbf{1}_{A_{c,n}} \right] \leq \frac{1}{a^j} \mathbf{E} \left[\left(\frac{n}{Y_0/a + \sum_{i=n/2}^n B_i} \right)^j \right].$$

We now show that

$$\limsup_{n \rightarrow \infty} \mathbf{E} \left[\left(\frac{n}{K_0 + \sum_{i=1}^n B_i} \right)^j \right] < \infty,$$

with $K_0 = Y_0/a$. To this end, we apply Theorem 2.1 of [12], with $n_0 = 1$, $p = j$, $Z_{i,n} = B_i + Y_0/n$ for $i \leq n$. All the assumptions of the theorem

are satisfied in our case. In fact, at first we have $\mathbf{E} [\bar{Z}_{n_0}^{-p}] < \infty$ because

$$\mathbf{E} [(Y_0 + B_1)^{-j}] \leq K_0^{-j} < \infty.$$

Secondly, note that $Z_{i,n}$ are identically distributed for all $i \leq n$, since B_i are i.i.d. Bernoulli of parameter c . Finally, \bar{Z}_n converges in distribution, since $\bar{Z}_n = \sum_{i=1}^n B_i/n + K_0 \xrightarrow{a.s.} c + K_0$. Hence, by Theorem 2.1 of [12], it follows that $\mathbf{E} [\bar{Z}_n^{-p}]$ is uniformly integrable. As a consequence,

$$\limsup_{n \rightarrow \infty} \mathbf{E} \left[\left(\frac{n}{K_0 + \sum_{i=1}^n B_i} \right)^j \right] = \limsup_{n \rightarrow \infty} \mathbf{E} [\bar{Z}_n^{-p}] < \infty.$$

Now, we will prove that $\mathbf{P}(A_{d,n})$ and $\mathbf{P}(A_{u,n})$ converge to zero exponentially fast. We will show that this occurs because $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ are bounded away from the extreme values $\{0; 1\}$, with probability that converge to one exponentially fast. Formally, fix $\epsilon > 0$, such that $\rho_1 + \epsilon < 1$ and $\rho_2 - \epsilon > 0$, and define $\alpha_n := n^\alpha$, $\alpha \in (0, 1)$, for any $n \geq 1$. Now, for any $n \geq 1$ define the following sets:

$$\begin{aligned} A_{1,n} &:= \left\{ \sup_{i \geq \alpha_n} \{\hat{\rho}_{1,i}\} > \rho_1 + \epsilon \right\}, \\ A_{2,n} &:= \left\{ \inf_{i \geq \alpha_n} \{\hat{\rho}_{2,i}\} < \rho_2 - \epsilon \right\}, \\ A_{3,n} &:= \left\{ \inf_{i \geq \alpha_n} \{\min\{1 - \hat{\rho}_{1,i}; \hat{\rho}_{2,i}\}\} \geq \min\{1 - \rho_1; \rho_2\} - \epsilon \right\}, \end{aligned}$$

where we recall that $\hat{\rho}_{1,i}$ and $\hat{\rho}_{2,i}$ are the adaptive thresholds. Note that $A_{1,n} \cup A_{2,n} \cup A_{3,n} = \Omega$. We have that

$$\begin{aligned} \mathbf{P}(A_{d,n}) &\leq \mathbf{P}(A_{1,n}) + \mathbf{P}(A_{2,n}) + \mathbf{P}(A_{3,n} \cap A_{d,n}), \\ \mathbf{P}(A_{u,n}) &\leq \mathbf{P}(A_{1,n}) + \mathbf{P}(A_{2,n}) + \mathbf{P}(A_{3,n} \cap A_{u,n}). \end{aligned}$$

First, we prove that $\mathbf{P}(A_{1,n})$ and $\mathbf{P}(A_{2,n})$ converge to zero exponentially fast. Consider the term $\mathbf{P}(A_{1,n})$. From the definition of $A_{1,n}$, we obtain

$$\mathbf{P}(A_{1,n}) = \mathbf{P} \left(\bigcup_{i \geq \alpha_n} \{\hat{\rho}_{1,i} > \rho_1 + \epsilon\} \right) \leq \sum_{i \geq \alpha_n} \mathbf{P}(\hat{\rho}_{1,i} > \rho_1 + \epsilon).$$

From (2.7), for large i we have that

$$\mathbf{P}(\hat{\rho}_{1,i} > \rho_1 + \epsilon) \leq c_1 \exp(-i\epsilon^2),$$

with $0 < c_1 < \infty$. Hence, using the fact that Y_n is increasing we have that

$$\begin{aligned} \mathbf{P}(A_{1,n}) &\leq \sum_{i \geq \alpha_n} \mathbf{P}(\hat{\rho}_{1,i} > \rho_1 + \epsilon) \\ &\leq c_1 \sum_{i \geq \alpha_n} \exp(-i\epsilon^2) \\ &= c_1 \exp(-\alpha_n \epsilon^2). \end{aligned}$$

Similar arguments can be applied to prove $\mathbf{P}(A_{2,n}) \rightarrow 0$ exponentially fast.

Finally, we show that $\mathbf{P}(A_{3,n} \cap A_{d,n})$ and $\mathbf{P}(A_{3,n} \cap A_{u,n})$ converge to zero exponentially fast. Consider $\mathbf{P}(A_{3,n} \cap A_{d,n})$, since the proof for $\mathbf{P}(A_{3,n} \cap A_{u,n})$ is analogous. First, let introduce $\phi := \min\{\rho_2; 1 - \rho_1\}$, and rewrite $A_{3,n}$ as follows:

$$A_{3,n} = \left\{ \inf_{i \geq \alpha_n} \{\hat{\rho}_{2,n}; 1 - \hat{\rho}_{1,n}\} \geq \phi - \epsilon \right\}.$$

Define the set $\tilde{A}_{d,n}$ as follows:

$$\tilde{A}_{d,n} := \left\{ \bigcap_{\alpha_n \leq i \leq n/2} \{Z_i < c\} \right\}.$$

We now set an appropriate value of c such that

$$(3.3) \quad \{A_{3,n} \cap A_{d,n}\} \subset \{A_{3,n} \cap \tilde{A}_{d,n}\},$$

for any $n \geq 1$. To do that, we need to set c such that $\{Z_i \geq c\} \subset \{Z_{i+1} \geq c\}$ for any $i \geq \alpha_n$. First, note that on the set $A_{3,n}$, $\{\hat{\rho}_{2,i} \geq (\phi - \epsilon)\}$ for any $i \geq \alpha_n$. Hence, for any $c < (\phi - \epsilon)$, if $\{c \leq Z_i \leq (\phi - \epsilon)\}$ we have $W_{2,i} = 0$, that implies $Z_{i+1} \geq Z_i$ and so $Z_{i+1} \geq c$. Alternatively, if $\{Z_i \geq (\phi - \epsilon) > c\}$, the set $\{Z_{i+1} \leq Z_i\}$ is possible, and hence we have to bound the increments of Z_n to guarantee that $Z_{i+1} \geq c$, i.e. set c such that

$$|Z_{i+1} - Z_i| < (\phi - \epsilon) - c, \quad \forall i \geq 0.$$

Using (3.1), we obtain

$$(3.4) \quad c \leq p_0 := \frac{Y_0}{Y_0 + b} \cdot (\phi - \epsilon).$$

This guarantees (3.3) holds for any $n \geq 1$.

We now show that $\mathbf{P}(A_{3,n} \cap \tilde{A}_{d,n})$ converges to zero exponentially fast. To this end, first note that on the set $A_{3,n}$, we have $\hat{\rho}_{2,i} > \rho_2 - \epsilon$ for any $i = \alpha_n, \dots, n/2$; moreover, on the set $\tilde{A}_{d,n}$, we have $Z_i < p_0$ for any $i = \alpha_n, \dots, n/2$. These considerations imply that $W_{2,i} = 0$ and $W_{1,i} = 1$ for any $i = \alpha_n, \dots, n/2$, on the set $A_{3,n} \cap \tilde{A}_{d,n}$. Hence, we can write

$$(3.5) \quad Z_{n/2} = \frac{Y_{1,\alpha_n} + \sum_{i=\alpha_n}^{n/2} X_i D_{1,i}}{Y_{\alpha_n} + \sum_{i=\alpha_n}^{n/2} X_i D_{1,i}} \geq \frac{y_{1,0} + a \sum_{i=\alpha_n}^{n/2} X_i}{(y_0 + \alpha_n b) + a \sum_{i=\alpha_n}^{n/2} X_i},$$

where the inequality is because $Y_{1,\alpha_n} \geq y_{1,0}$, $Y_{\alpha_n} \leq y_0 + \alpha_n b$ and $D_{1,i} \geq a$ a.s. for any $i \geq 1$. Now, define for any $n \geq 1$ the set $A_{4,n}$ as follows:

$$A_{4,n} := \left\{ \sum_{i=\alpha_n}^{n/2} X_i > \frac{p_0}{a(1-p_0)} (y_0 + \alpha_n b) \right\},$$

and consider the set $A_{\phi,n} \cap \tilde{A}_{d,n} \cap A_{4,n}$. On the set $A_{\phi,n} \cap \tilde{A}_{d,n}$ we can use the definition of $A_{4,n}$ in (3.5), obtaining

$$\{A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}\} \subset \{Z_{n/2} > p_0\} \cap \tilde{A}_{d,n}.$$

However, $\{Z_{n/2} > p_0\} \cap \tilde{A}_{d,n} = \emptyset$. Hence, $\mathbf{P}(A_{\phi,n} \cap \tilde{A}_{d,n} \cap A_{4,n}) = 0$ and it is sufficient to show that $\mathbf{P}(A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}^C)$ converges to zero exponentially fast.

To this end, note that on the set $A_{3,n} \cap \tilde{A}_{d,n}$ we have $Z_{i+1} \geq Z_i$ for any $i = \alpha_n, \dots, n/2$, since we previously showed that $W_{2,i} = 0$ and $W_{1,i} = 1$. Hence, on the set $A_{3,n} \cap \tilde{A}_{d,n}$, $\{X_i, i = \alpha_n, \dots, n/2\}$ are conditionally Bernoulli with parameter $p_i \geq Z_{\alpha_n}$ a.s. Now, let denote with $\{\varrho_{i,n}; i = 1, \dots, n/2 - \alpha_n\}$ a sequence of i.i.d. Bernoulli random variable with parameter $z_{0,n}$, defined as

$$z_{0,n} := \frac{y_{1,0}}{y_0 + \alpha_n b} \leq Z_{\alpha_n} \quad a.s.;$$

it follows that $\mathbf{P}(A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}^C)$ is less than or equal than

$$(3.6) \quad \mathbf{P} \left(\sum_{i=1}^{n/2-\alpha_n} \varrho_{i,n} \leq \frac{p_0}{a(1-p_0)} (y_0 + \alpha_n b) \right).$$

Finally, we use the following Chernoff's upper bound for i.i.d. random variables in $[0, 1]$ (see [10])

$$(3.7) \quad \mathbf{P}(S_n \leq c_0 \cdot \mathbf{E}[S_n]) \leq \exp \left(-\frac{(1-c_0)^2}{2} \cdot \mathbf{E}[S_n] \right),$$

with $c_0 \in (0, 1)$ and $S_n = \sum_i^n X_i$. In our case, we have that (3.6) can be written as $\mathbf{P}(S_n \leq c_n \cdot \mathbf{E}[S_n])$, where $S_n = \sum_{i=1}^{n/2-\alpha_n} \varrho_{i,n}$ and

$$\mathbf{E}[S_n] = \left(\frac{n}{2} - \alpha_n\right) \frac{y_{1,0}}{(y_0 + \alpha_n b)} \quad \text{and} \quad c_n = \frac{p_0}{a(1-p_0)} \frac{(y_0 + \alpha_n b)^2}{y_{1,0}(n/2 - \alpha_n)};$$

since $c_n \rightarrow 0$, we can define an integer n_0 such that $c_n < c_0$ for any $n \geq n_0$, so that

$$\mathbf{P}(S_n \leq c_n \cdot \mathbf{E}[S_n]) \leq \mathbf{P}(S_n \leq c_0 \cdot \mathbf{E}[S_n]).$$

Hence, by using (3.7), for any $n \geq n_0$ we have that

$$\mathbf{P}(A_{3,n} \cap A_{4,n}^C) \leq \exp\left(-\frac{(1-c_0)^2}{2} \cdot \mathbf{E}[S_n]\right),$$

which converges to zero exponentially fast since

$$\mathbf{E}[S_n] = \frac{y_{1,0}(n/2 - \alpha_n)}{y_0 + \alpha_n b} \sim \frac{n}{\alpha_n} = n^{1-\alpha}.$$

This concludes the proof. \square

REMARK 3.1. *The result of Theorem 3.1 can be also obtained relaxing assumption (2.7). In that case, we need condition (2.2) and (c2) to be satisfied. Then, the proof is the same by setting*

$$\begin{aligned} A_{1,n} &:= \left\{ \sup_{i \geq \alpha_n} \{\hat{\rho}_{1,i}\} > 1 - \epsilon \right\}, \\ A_{2,n} &:= \left\{ \inf_{i \geq \alpha_n} \{\hat{\rho}_{2,i}\} < \epsilon \right\}, \\ A_{3,n} &:= \left\{ \inf_{i \geq \alpha_n} \{\min\{1 - \hat{\rho}_{1,i}; \hat{\rho}_{2,i}\}\} \geq \epsilon \right\}, \end{aligned}$$

where $0 < \epsilon < 1/2$ is such that $\hat{\rho}_{j,n} \in [\epsilon, 1 - \epsilon]$ for any $n \geq 1$ and $j = 1, 2$. Then, $\mathbf{P}(A_{1,n}) = \mathbf{P}(A_{2,n}) = 0$ for any $n \geq 1$.

3.2. *A uniform bound.* In this subsection, we provide a uniform bound for the scaled difference between Z_t and $\tilde{\rho}_t$ (which is \mathcal{F}_{t-} -measurable). To make precise statements, we start by defining some notations. Set $\Delta_{j,k} := \text{sign}(m_1 - m_2) (\tilde{\rho}_{q^j+k} - Z_{q^j+k})$ and $\tilde{T}_{j,k} := Y_{q^j+k} \Delta_{j,k}$, for any $j \geq 1$ and any $k = 1, \dots, d_j$, where $d_j := q^{j+1} - q^j$. Note that, since from (2.5) $\tilde{\rho}_{1,q^j+k} = \hat{\rho}_{1,q^j}$

for any $k \in \{1, \dots, d_j\}$, we can also write $\Delta_{j,k} = \text{sign}(m_1 - m_2) (\hat{\rho}_{q^j} - Z_{q^j+k})$. Let $\{\tau_j; j \geq 1\}$ be a sequence of stopping times defined as follows:

$$(3.8) \quad \tau_j := \begin{cases} \inf \left\{ k \geq 1 : \tilde{T}_{j,k} \in [-b, 0] \right\} & \text{if } \left\{ k \geq 1 : \tilde{T}_{j,k} \in [-b, 0] \right\} \neq \emptyset; \\ \infty & \text{otherwise.} \end{cases}$$

In Theorem 3.2 we provide a L_1 -uniform bound for the scaled distance among urn proportion Z_{q^j+k} and the threshold $\tilde{\rho}_{q^j+k}$, on the set $\{\tau_j \leq k\}$.

THEOREM 3.2. *Let $\tilde{\rho}_{1,n}$ and $\tilde{\rho}_{2,n}$ be as in (2.5). Then, under assumption (2.1) and (2.7), there exists a constant $C > 0$ such that*

$$(3.9) \quad \sup_{j \geq 1} \sup_{1 \leq k \leq d_j} \mathbf{E} \left[q^j \cdot |\Delta_{j,k}| \mathbf{1}_{\{\tau_j \leq k\}} \right] < C,$$

where $d_j = q^{j+1} - q^j$.

The proof uses comparison arguments with the MRRU model and related asymptotic results. Hence, we first present the results concerning the MRRU model in Subsection 3.2.1. The proof of Theorem 3.2 is reported in Subsection 3.2.2.

3.2.1. Estimates for the MRRU model. In this subsection, we present some probabilistic estimates concerning the MRRU model which are needed in the proof of Theorem 3.2. We recall that for the MRRU the threshold are fixed, i.e. $\hat{\rho}_{j,n} = \rho_j$ for any $n \geq 1$, $j = 1, 2$. Hence, in this subsection we consider $W_{1,n} = \mathbf{1}_{\{Z_n \leq \rho_1\}}$ and $W_{2,n} = \mathbf{1}_{\{Z_n \geq \rho_2\}}$. We start by introducing some quantities related to the MRRU model. Let $\{T_n; n \geq 0\}$ be the process defined as

$$(3.10) \quad T_n := \text{sign}(m_1 - m_2) \cdot Y_n(\rho - Z_n),$$

which is sometimes useful to represent it as follows:

$$T_n = \text{sign}(m_1 - m_2) \cdot (\rho Y_{2,n} - (1 - \rho) Y_{1,n}).$$

Then, let t_0 be the following stopping time

$$(3.11) \quad t_0 := \inf \{ k \geq 0 : T_k \in [-b, 0] \}.$$

Let

$$S_n := \{0 \leq k \leq n : T_{n-k} \in [-b, 0]\},$$

and let $\{s_n; n \geq 1\}$ be a sequence of random times defined as

$$(3.12) \quad s_n = \begin{cases} \inf\{S_n\} & \text{if } S_n \neq \emptyset; \\ \infty & \text{otherwise.} \end{cases}$$

where we recall that b is the maximum value of the urn reinforcements, i.e. $D_{1,n}, D_{2,n} \leq b$ a.s. for any $n \geq 1$. Note that by definition $\{s_n = \infty\} = \{t_0 > n\}$. In Theorem 3.3 we provide the L_2 -uniform bound for $Y_n(Z_n - \rho)$, on the set $\{t_0 \leq n\}$.

THEOREM 3.3. *For an MRRU, under assumption (2.1), there exists a constant $C > 0$ such that*

$$(3.13) \quad \sup_{n \geq 1} \mathbf{E} \left[(Y_n | \rho - Z_n)^2 \mid t_0 \leq n \right] \leq C.$$

The proof uses the boundedness of the moments of the excursion times s_n , which is provided in Theorem 3.4. Hence, we first present Theorem 3.4 and then we report the proof of Theorem 3.3.

THEOREM 3.4. *For an MRRU, under assumption (2.1), there exists a constant $C > 0$ such that*

$$\sup_{n \geq 1} \{ \mathbf{E} [s_n^2 | t_0 \leq n] \} \leq C.$$

In the proof of Theorem 3.4, we need to couple the MRRU model with a particular urn model $\{\tilde{Z}_n; n \geq 1\}$. The processes are coupled, in the sense that: (i) the potential reinforcements are the same, i.e. $\tilde{D}_{1,n} = D_{1,n}$ and $\tilde{D}_{2,n} = D_{2,n}$ a.s.; (ii) the drawing process is defined on the same probability space, i.e. $\tilde{U}_n = U_n$ a.s. where $\{U_n; n \geq 1\}$ and $\{\tilde{U}_n; n \geq 1\}$ are i.i.d. uniform random variables such that $X_{n+1} := \mathbf{1}_{\{U_{n+1} < Z_n\}}$ and $\tilde{X}_{n+1} := \mathbf{1}_{\{\tilde{U}_{n+1} < \tilde{Z}_n\}}$ for any $n \geq 1$, respectively.

We now describe the urn model $\{\tilde{Z}_n; n \geq 1\}$. Fix a constant $\tilde{y}_0 \in (0, Y_0]$ and $z_0 = \rho_1$. The process $\{\tilde{Z}_n; n \geq 1\}$ evolves as follows:

if $s_{n-1} = 0$, i.e. $Z_{n-1} \geq \rho_1$, then $\tilde{X}_n = \mathbf{1}_{\{\tilde{U}_n < \rho_1\}}$ and

$$(3.14) \quad \begin{cases} \tilde{Y}_{1,n} = \rho_1 \cdot \tilde{y}_0 + \tilde{X}_n \tilde{D}_{1,n}, \\ \tilde{Y}_{2,n} = (1 - \rho_1) \cdot \tilde{y}_0 + (1 - \tilde{X}_n) \tilde{D}_{2,n}; \end{cases}$$

if $s_{n-1} = k \geq 1$, i.e. $Z_{n-1} < \rho_1$, then $\tilde{X}_n = \mathbf{1}_{\{\tilde{U}_n < \tilde{Z}_{n-1}\}}$ and

$$(3.15) \quad \begin{cases} \tilde{Y}_{1,n} = \tilde{Y}_{1,n-1} + \tilde{X}_n \tilde{D}_{1,n}, \\ \tilde{Y}_{2,n} = \tilde{Y}_{2,n-1} + (1 - \tilde{X}_n) \tilde{D}_{2,n}; \end{cases}$$

where $\tilde{Y}_n := \tilde{Y}_{1,n} + \tilde{Y}_{2,n}$ and $\tilde{Z}_n := \tilde{Y}_{1,n}/\tilde{Y}_n$. The urn model is well defined since s_{n-1} is \mathcal{F}_{n-1} -measurable. It is worth noticing that \tilde{Z}_n represents a Generalized Pólya urn evaluated after exactly $(s_{n-1} + 1)$ steps, with initial composition $\rho_1 \tilde{y}_0$ red and $\rho_1 (1 - \tilde{y}_0)$ white balls.

In the next lemma, we state an important relation among the MRRU model and the process $\{\tilde{Z}_n; n \geq 1\}$, needed in the proof of Theorem 3.4.

LEMMA 3.2. *Consider the urn model $\{\tilde{Z}_n; n \geq 1\}$ defined in (3.14) and (3.15) coupled with the MRRU process $\{Z_n; n \geq 1\}$. Let $\tilde{T}_n := \text{sign}(m_1 - m_2) \cdot \tilde{Y}_n (\rho - \tilde{Z}_n)$ for any $n \geq 1$. Then, on the set $\{\exists j < n : T_j \leq 0\}$, we have that*

$$\{T_n > 0\} \subset \{\tilde{T}_n \geq T_n\}.$$

PROOF. Wlog assume $m_1 > m_2$, which implies $\rho = \rho_1$ and $T_n = Y_n (\rho_1 - Z_n)$. Sometimes, we will prefer the following expression of \tilde{T}_n

$$\tilde{T}_n = \rho_1 \tilde{Y}_{2,n} - (1 - \rho_1) \tilde{Y}_{1,n}.$$

The proof will be by induction. Note that, on the set $\{\exists j < n : T_j \leq 0\}$, s_n is almost surely finite. On the set $\{s_n = 0\}$, i.e. $\{T_n \leq 0\}$, we can immediately show that $\{T_{n+1} > 0\}$ implies $\{\tilde{T}_{n+1} \geq T_{n+1}\}$ and $\{\tilde{Z}_{n+1} \leq Z_{n+1}\}$. In fact, from $\{T_n \leq 0\}$ and $\{T_{n+1} > 0\}$ we have $X_{n+1} = 0$ and $W_{2,n} = 1$, so that

$$T_{n+1} = T_n + \rho_1 D_{2,n+1} \leq \rho_1 D_{2,n+1} = \rho_1 \tilde{D}_{2,n+1} = \tilde{T}_{n+1}$$

and

$$Z_{n+1} = \frac{Z_n Y_n}{Y_n + D_{2,n+1}} \geq \frac{\rho_1 \tilde{y}_0}{\tilde{y}_0 + \tilde{D}_{2,n+1}} = \tilde{Z}_{n+1}.$$

Now, consider the set $\{s_n \geq 1\}$ and assume by induction hypothesis that

$$(3.16) \quad \left\{ \tilde{T}_i \geq T_i > 0, \tilde{Z}_i \leq Z_i < \rho_1, \forall i = n - s_n + 1, \dots, n \right\} \cap \{T_{n+1} > 0\}.$$

Then, we will show that $\tilde{T}_{n+1} \geq T_{n+1}$ and $\tilde{Z}_{n+1} \leq Z_{n+1}$. Since $T_n = \rho_1 Y_{2,n} - (1 - \rho_1) Y_{1,n}$, we note that

$$T_{n+1} = T_{n-s_n} + \sum_{i=n-s_n+1}^{n+1} [\rho_1 (1 - X_i) D_{2,i} W_{2,i-1} - (1 - \rho_1) X_i D_{1,i} W_{1,i-1}],$$

where we recall that for the MRRU model $W_{1,i} = \mathbf{1}_{\{Z_i \leq \rho_1\}}$ and $W_{2,i} = \mathbf{1}_{\{Z_i \geq \rho_2\}}$. Since $T_{n-s_n} \leq 0$, $W_{2,i-1} \leq 1$, by (3.16) $W_{1,i} = 1$ for any $i = n - s_n + 1, \dots, n$, $X_{n-s_n+1} = 0$, and by construction $\tilde{D}_{1,i} = D_{1,i}$ and $\tilde{D}_{2,i} = D_{2,i}$, we have

$$T_{n+1} \leq \sum_{i=n-s_n+1}^{n+1} [\rho_1 (1 - X_i) \tilde{D}_{2,i} - (1 - \rho_1) X_i \tilde{D}_{1,i}].$$

Moreover, by (3.16) we have $X_{i+1} = \mathbf{1}_{\{U_{i+1} < Z_i\}} \geq \mathbf{1}_{\{\tilde{U}_{i+1} < \tilde{Z}_i\}} = \tilde{X}_{i+1}$ for any $i = n - s_n + 1, \dots, n$. Hence, we can write

$$T_{n+1} \leq \rho_1 \sum_{i=n-s_n+1}^{n+1} (1 - \tilde{X}_i) \tilde{D}_{2,i} - (1 - \rho_1) \sum_{i=n-s_n+1}^{n+1} \tilde{X}_i \tilde{D}_{1,i} = \tilde{T}_{n+1}.$$

Similarly, we can prove that $\tilde{Z}_{n+1} \leq Z_{n+1}$. Note that

$$Z_{n+1} = \frac{Z_{n-s_n} Y_{n-s_n} + \sum_{i=n-s_n+1}^{n+1} X_i D_{1,i} W_{1,i-1}}{Y_{n-s_n} + \sum_{i=n-s_n+1}^{n+1} X_i D_{1,i} W_{1,i-1} + \sum_{i=n-s_n+1}^{n+1} (1 - X_i) D_{2,i} W_{2,i-1}}.$$

Now, since $Z_{n-s_n} \geq \rho_1$, $Y_{n-s_n} \geq \tilde{y}_0$ and $X_{i+1} \geq \tilde{X}_{i+1}$ for any $i = n - s_n + 1, \dots, n$, it follows that

$$Z_{n+1} \geq \frac{\rho_1 Y_0 + \sum_{i=n-s_n+1}^{n+1} \tilde{X}_i \tilde{D}_{1,i}}{\tilde{y}_0 + \sum_{i=n-s_n+1}^{n+1} \tilde{X}_i \tilde{D}_{1,i} + \sum_{i=n-s_n+1}^{n+1} (1 - \tilde{X}_i) \tilde{D}_{2,i}} = \tilde{Z}_{n+1},$$

which concludes our proof by induction. \square

PROOF OF THEOREM 3.4. Wlog assume $m_1 > m_2$, which implies $\rho = \rho_1$ and $T_n = Y_n(\rho_1 - Z_n)$. The structure of the proof is the following. The aim is to show that $\mathbf{P}(s_n = k | t_0 \leq n)$ converges to zero fast enough such that $\mathbf{E}[s_n^2 | t_0 \leq n]$ is bounded. To this end, we consider the urn model $\{\tilde{Z}_n; n \geq 1\}$ defined in (3.14) and (3.15) coupled with the MRRU model, such that $\mathbf{P}(s_n = k | t_0 \leq n)$ can be expressed in terms of $\{\tilde{Z}_n; n \geq 1\}$. After some calculations, this is provided by Lemma 3.2. Moreover, we compare

$\{\tilde{Z}_n; n \geq 1\}$ with a Generalized Pólya urn model, whose moments are uniformly bounded.

First, for any $n \geq 1$ note that

$$\mathbf{E} [s_n^2 | t_0 \leq n] = \sum_{k=1}^n k^2 \mathbf{P}(s_n = k | t_0 \leq n),$$

since $\mathbf{P}(s_n = \infty | t_0 \leq n) = \mathbf{P}(t_0 > n | t_0 \leq n) = 0$. In fact, by definition $t_0 \leq n - s_n$ a.s.

Before considering the urn model $\{\tilde{Z}_n; n \geq 1\}$, we express $\mathbf{P}(s_n = k | t_0 \leq n)$ in terms of $\{T_n; n \geq 1\}$. Note that in the MRRU, if $T_j \geq -b$ for some $j \geq 0$, then $\mathbf{P}(T_n < -b) = 0$ for any $n \geq j$. In fact, when $T_n \geq 0$ ($Z_n \leq \rho_1$) we have $T_{n+1} \geq -b$, because the reinforcements are bounded by b and so $|T_{n+1} - T_n| < b$ a.s.; while when $-b \leq T_n < 0$ ($Z_n > \rho_1$) we have $T_{n+1} \geq T_n \geq -b$, because $Z_n > \rho_1$ implies $W_{1,n} = 0$ and so the urn is not reinforced by red balls, i.e. $T_{n+1} \geq T_n$. As a consequence, since $T_{t_0} \geq -b$ by definition, on the set $\{n \geq t_0\}$, we have $\{T_n \notin [-b, 0]\} \subset \{T_n > 0\}$. Hence, since $t_0 \leq n - s_n$, we have for all $1 \leq k \leq n$

(3.17)

$$\begin{aligned} \mathbf{P}(s_n = k | t_0 \leq n) &= \mathbf{P}\left(\cap_{i=0}^{k-1} \{T_{n-i} > 0\} \cap \{T_{n-k} \leq 0\} | t_0 \leq n\right) \\ &\leq \mathbf{P}\left(\cap_{i=0}^{k-1} \{T_{n-i} > 0\} | \{T_{n-k} \leq 0\} \cap \{t_0 \leq n\}\right) \\ &\leq \mathbf{P}\left(\cap_{i=0}^{k-1} \{T_{n-i} > 0\} | T_{n-k} \leq 0\right), \end{aligned}$$

where the last inequality follows from $\{T_{n-k} \leq 0\} \subseteq \{t_0 \leq n\}$. To deal with (3.17), we consider the urn model $\{\tilde{Z}_n; n \geq 1\}$ defined in (3.14) and (3.15). From Lemma 3.2, we have that, on the set $\{\exists j < n : T_j \leq 0\}$, the event $\{T_n > 0\}$ implies $\{\tilde{T}_n \geq T_n\}$. Hence, we have that

(3.18)

$$\begin{aligned} \mathbf{P}\left(\cap_{i=0}^{k-1} \{T_{n-i} > 0\} | T_{n-k} \leq 0\right) &\leq \mathbf{P}\left(\cap_{i=0}^{k-1} \{\tilde{T}_{n-i} > 0\} | T_{n-k} \leq 0\right) \\ &= \mathbf{P}\left(\cap_{i=1}^k \{Z_i^G < \rho_1\}\right), \end{aligned}$$

by construction, where $\{Z_i^G; i \geq 1\}$ is the proportion of red balls of a Generalized Pólya urn, starting with a proportion of $Z_0^G = \rho_1$ and an initial number of balls $Y_0^G = \tilde{y}_0$, and the same reinforcements distributions as $D_{1,n}$ and $D_{2,n}$.

Now, let s^G be the first time the process Z_i^G is above ρ_1 , i.e.

$$s^G := \begin{cases} \inf \{ i \geq 1 : Z_i^G \geq \rho_1 \} & \text{if } \{ i \geq 1 : Z_i^G \geq \rho_1 \} \neq \emptyset; \\ \infty & \text{otherwise.} \end{cases}$$

It can be shown using standard arguments that there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, there exist $0 < c_1, c_2 < \infty$

$$\mathbf{P}(s^G = k) \leq c_1 \exp(-c_2 k),$$

which implies that $\mathbf{E}[\exp(\gamma s^G)] < \infty$ for some $\gamma > 0$.

Now, returning to (3.18), we have that

$$\mathbf{P}\left(\cap_{i=1}^k \{Z_i^G < \rho_1\}\right) = \mathbf{P}(s^G > k) \leq \frac{\mathbf{E}[(s^G)^4]}{k^4} = \frac{C_4}{k^4}.$$

Thus we have for any $k \geq 1$

$$\mathbf{P}(s_n = k | t_0 \leq n) \leq \frac{C_4}{k^4},$$

and hence

$$\begin{aligned} \mathbf{E}[s_n^2 | t_0 \leq n] &= \sum_{k=1}^n k^2 \mathbf{P}(s_n = k | t_0 \leq n) \\ &\leq C_4 \cdot \sum_{k=1}^n \frac{1}{k^2} < C < \infty. \end{aligned}$$

This concludes the proof. \square

PROOF OF THEOREM 3.3. Wlog assume $m_1 > m_2$, which implies $\rho = \rho_1$. Since $T_n = Y_n(\rho_1 - Z_n)$ we want to prove

$$\sup_{n \geq 1} \mathbf{E}[T_n^2 | t_0 \leq n] < \infty.$$

Let s_n be the random time defined in (3.12). Then, since $|T_{i+1} - T_i| \leq b$ a.s. for any $i \geq 1$ and from (3.12) $T_{n-s_n} \in [-b, 0]$, we have

$$\begin{aligned} \mathbf{E}[T_n^2 | t_0 \leq n] &= \sum_{l=0}^n \mathbf{E}[T_n^2 | \{s_n = l\} \cap \{t_0 \leq n\}] \mathbf{P}(s_n = l | t_0 \leq n) \\ &= b^2 + \sum_{l=1}^n \mathbf{E}\left[\left(\sum_{i=n-l}^{n-1} (T_{i+1} - T_i) + T_{n-l}\right)^2 | \{s_n = l\}\right] \mathbf{P}(s_n = l | t_0 \leq n) \\ &\leq \sum_{l=0}^n (l+1)^2 b^2 \mathbf{P}(s_n = l | t_0 \leq n). \end{aligned}$$

Now, using $(l+1)^2 \leq 4l^2$, we have that

$$\sum_{l=0}^n (l+1)^2 b^2 \mathbf{P}(s_n = l | t_0 \leq n) \leq 4b^2 \cdot \mathbf{E}[s_n^2 | t_0 \leq n].$$

Finally, using Theorem 3.4 we have that the last quantity is uniformly bounded by a constant C independent of n , so the proof is concluded. \square

REMARK 3.2. *From the proof of Theorem 3.4, we have that the constant C is independent of the initial proportion Z_0 . Moreover, C provides a uniform bound for any other MRRU with initial number of balls $\geq Y_0$.*

3.2.2. Proof of Theorem 3.2.

PROOF. Wlog, assume $m_1 > m_2$, which implies $\rho = \rho_1$. First, fix $j \in \mathbb{N}$ and apply Cauchy-Schwarz, so obtaining

$$\left(\mathbf{E} \left[q^j \cdot |\Delta_{j,k}| \mathbf{1}_{\{\tau_j \leq k\}} \right] \right)^2 \leq \mathbf{E} \left[\left(\tilde{T}_{j,k} \right)^2 \mathbf{1}_{\{\tau_j \leq k\}} \right] \mathbf{E} \left[\left(\frac{q^j}{Y_{q^j}} \right)^2 \right].$$

Since $\mathbf{E} \left[\left(\frac{q^j}{Y_{q^j}} \right)^2 \right]$ is uniformly bounded by Theorem 3.1, it remains to prove that

$$\mathbf{E} \left[\left(\tilde{T}_{j,k} \right)^2 \mathbf{1}_{\{\tau_j \leq k\}} \right] < C,$$

for any $j \geq 1$ and any $k = 1, \dots, d_j$. To this end, fix $j \in \mathbb{N}$ and note that since $\tilde{\rho}_{1,q^j+k} = \hat{\rho}_{1,q^j}$ for any $k \in \{1, \dots, d_j\}$, the process $\{Z_{q^j+k}; k = 1, \dots, d_j\}$ can be considered as the urn proportion of the MRRU model, with initial composition (Y_{1,q^j}, Y_{2,q^j}) and fixed threshold parameters $\hat{\rho}_{1,q^j}$ and $\hat{\rho}_{2,q^j}$. Then, for each $j \in \mathbb{N}$ we can apply Theorem 3.3, with t_0 defined in (3.11) equal to τ_j , so obtaining

$$(3.19) \quad \mathbf{E} \left[\left(\tilde{T}_{j,k} \right)^2 \mathbf{1}_{\{\tau_j \leq k\}} \right] \leq C_j,$$

where C_j is a constant depending on the initial composition (Y_{1,q^j}, Y_{2,q^j}) . However, from Remark 3.2 we have that there exists a uniform bound $C > 0$ such that $C_j \leq C$ for any $j \geq 1$, since all the processes $\{Z_{q^j+k}, k = 1, \dots, d_j\}$ $j \geq 1$ can be considered as MRRU with initial number of balls $\geq Y_0$; this concludes the proof. \square

4. Proofs of the main results. Here, we present the proofs of the results described in Section 2. Subsection 4.1 is dedicated to the proof of Theorem 2.1 (LLN) and the related preliminary results. Then, in subsection 4.2 we report the proof of Theorem 2.2 (CLT) together with Theorem 4.1, a new result needed to compute that proof. In the last subsections, the proofs of the remaining results of Section 2 are gathered.

4.1. *Proof of the LLN.* We start by reporting some preliminary results needed in the proof of the LLN. Initially, we show that the number of balls sampled from the urn $N_{1,n}$, $N_{2,n}$ and the total number of balls in the urn Y_n , increase to infinity almost surely. To do that, we first need to show a lower bound for the increments of the process Y_n , which is given by the following:

LEMMA 4.1. *For any $i \geq 1$, we have that*

$$\mathbf{E}[Y_i - Y_{i-1} | \mathcal{F}_{i-1}] \geq a \cdot \left(\frac{\min\{y_{1,0}; y_{2,0}\}}{y_0 + (i-1)b} \right).$$

PROOF. First, note that

$$Y_i - Y_{i-1} = X_i D_{1,i} W_{1,i-1} + (1 - X_i) D_{2,i} W_{2,i-1}.$$

Since X_i and $D_{1,i}$ are conditionally independent with respect to \mathcal{F}_{i-1} , and $W_{1,i-1}$ is \mathcal{F}_{i-1} -measurable, we have that

$$\begin{aligned} \mathbf{E}[Y_i - Y_{i-1} | \mathcal{F}_{i-1}] &= (m_1 Z_{i-1} W_{1,i-1} + m_2 (1 - Z_{i-1}) W_{2,i-1}) \\ &\geq a \cdot (Z_{i-1} W_{1,i-1} + (1 - Z_{i-1}) W_{2,i-1}), \end{aligned}$$

where the last inequality is because $m_1, m_2 \geq a$. We recall that the variables $W_{1,i-1}$ and $W_{2,i-1}$ can only take the values 0 and 1, and by construction we have that $W_{1,i-1} + W_{2,i-1} \geq 1$ for any $i \geq 1$; then, we can give a further lower bound

$$(4.1) \quad \mathbf{E}[Y_i - Y_{i-1} | \mathcal{F}_{i-1}] \geq a \cdot (\min\{Z_{i-1}; 1 - Z_{i-1}\}).$$

Finally, the result follows by noting that

$$\min\{Z_{i-1}; 1 - Z_{i-1}\} = \frac{\min\{Y_{1,i-1}; Y_{2,i-1}\}}{Y_{i-1}} \geq \frac{\min\{y_{1,0}; y_{2,0}\}}{y_0 + (i-1)b}.$$

□

Here, we present the lemma on the divergence of the sequences Y_n , $N_{1,n}$ and $N_{2,n}$. This result is obtained by using the conditional Borel-Cantelli lemma.

LEMMA 4.2. *Consider the urn model presented in Section 2. Then,*

- (a) $Y_n \xrightarrow{a.s.} \infty$;
- (b) $\min\{N_{1,n}; N_{2,n}\} \xrightarrow{a.s.} \infty$.

PROOF. We begin with the proof of part (a). First, notice that $Y_n = \sum_{i=1}^n (Y_i - Y_{i-1}) + y_0$. Then, by Theorem 1 in [9], it is sufficient to show that

$$\left\{ \omega \in \Omega : \sum_{i=1}^{\infty} [Y_i - Y_{i-1} | \mathcal{F}_{i-1}] = \infty \right\}$$

occurs with probability one. To this end, we will now use the lower bound of Lemma 4.1, so obtaining

$$\sum_{i=1}^n \mathbf{E} [Y_i - Y_{i-1} | \mathcal{F}_{i-1}] \geq a \left(\sum_{i=1}^n \frac{\min\{y_{1,0}; y_{2,0}\}}{y_0 + (i-1)b} \right) \xrightarrow{a.s.} \infty.$$

Hence, we have that $Y_n \xrightarrow{a.s.} \infty$.

We now report the proof of part (b). We will show that $N_{1,n} \xrightarrow{a.s.} \infty$, since the proof for $N_{2,n}$ is analogous. Since $N_{1,n} = \sum_{i=1}^n X_i$, by Theorem 1 in [9], it is sufficient to show that

$$\left\{ \omega \in \Omega : \sum_{i=1}^{\infty} \mathbf{P}(X_i | \mathcal{F}_{i-1}) = \infty \right\}$$

occurs with probability one. Then, we obtain

$$\sum_{i=1}^n \mathbf{P}(X_i | \mathcal{F}_{i-1}) = \sum_{i=1}^n Z_i \geq \sum_{i=1}^n \frac{y_{1,0}}{y_0 + (i-1)b} \xrightarrow{a.s.} \infty.$$

Hence, we have that $N_{1,n} \xrightarrow{a.s.} \infty$. \square

The following lemma corresponds to Theorem 2.1 of [3], and it is needed in the proof of Theorem 2.1. This result provides multiple equivalent ways to show the almost sure convergence of a real-valued process. We consider a general real-valued process $\{Z_n; n \geq 0\}$ and two real numbers d (down) and u (up), with $d < u$. The result requires two sequences of times $t_j(d, u)$ and $\tau_j(d, u)$ defined as follows: for each $j \geq 0$, $t_j(d, u)$ represents the time of the first up-cross of u after $\tau_{j-1}(d, u)$, and $\tau_j(d, u)$ represents the time of the first down-cross of d after t_j . Note that $t_j(d, u)$ and $\tau_j(d, u)$ are stopping times, since the events $\{t_j(d, u) = k\}$ and $\{\tau_j(d, u) = k\}$ depend on $\{Z_n; n \leq k\}$, which are measurable with respect to \mathcal{F}_k . We omit the proof since it is reported in Theorem 2.1 of [3], using the same notation.

LEMMA 4.3. *Let $\{Z_n; n \geq 0\}$ be a real-valued process in $[0, 1]$. Let $\tau_{-1}(d, u) = -1$ and define for every $j \geq 0$ two stopping times*

$$(4.2) \quad \begin{aligned} t_j(d, u) &= \begin{cases} \inf\{n > \tau_{j-1}(d, u) : Z_n > u\} & \text{if } \{n > \tau_j(d, u) : Z_n > u\} \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases} \\ \tau_j(d, u) &= \begin{cases} \inf\{n > t_j(d, u) : Z_n < d\} & \text{if } \{n > t_{j-1}(d, u) : Z_n < d\} \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Then, the following three events are a.s. equivalent

- (a) Z_n converges a.s.;
- (b) for any $0 < d < u < 1$,

$$\lim_{j \rightarrow \infty} \mathbf{P}(t_j(d, u) < \infty) = 0;$$

- (c) for any $0 < d < u < 1$,

$$\sum_{j \geq 1} \mathbf{P}(t_{j+1}(d, u) = \infty | t_j(d, u) < \infty) = \infty;$$

using the convention that $\mathbf{P}(t_{j+1}(d, u) = \infty | t_j(d, u) < \infty) = 1$ when $\mathbf{P}(t_j(d, u) = \infty) = 1$.

The following lemma provides lower bounds for the total number of balls in the urn at the times of up-crossings Y_{t_j} . The lemma gets used in the proof of Theorem 2.1, where conditioning to a fixed number of up-crossing ensures to have at least a number of balls Y_n determined by the lower bounds of this lemma. This result has been taken by Lemma 2.1 of [3]. We omit the proof since adaptive thresholds does not play any role during up-crossings and the proof reported in Lemma 2.1 of [3] carries over to our model, with D_n replaced by Y_n .

LEMMA 4.4. *For any $0 < d < u < 1$, we have that*

$$Y_{t_j(d, u)} \geq \left(\frac{u(1-d)}{d(1-u)} \right) Y_{t_{j-1}(d, u)} \geq \dots \geq \left(\frac{u(1-d)}{d(1-u)} \right)^j Y_{t_0(d, u)}.$$

The following lemma provides a uniform bound for the generalized Pólya urn with same reinforcement means, which is needed in the proof of Theorem 2.1. This result has been taken from Lemma 3.2 of [3]. The proof is omitted since it is reported in [3].

LEMMA 4.5. *Consider a generalized Pólya urn with $m_1 = m_2$. If $Y_0 \geq 2b$, then*

$$\mathbf{P} \left(\sup_{n \geq 1} |Z_n - Z_0| \geq h \right) \leq \frac{b}{Y_0} \left(\frac{4}{h^2} + \frac{2}{h} \right)$$

for every $h > 0$.

Here, we provide the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. Wlog assume $m_1 > m_2$, which implies $\hat{\rho}_n = \hat{\rho}_{1,n}$ and $\rho = \rho_1$. We divide the proof in two steps:

- (a) $\mathbf{P} \left(\limsup_{n \rightarrow \infty} Z_n = \rho_1 \right) = 1$,
- (b) $\mathbf{P} \left(\lim_{n \rightarrow \infty} Z_n \text{ exists} \right) = 1$.

Proof of part (a):

We begin by proving that $\mathbf{P}(\limsup_{n \rightarrow \infty} Z_n \leq \rho_1) = 1$. To this end, we show that there cannot exist $\epsilon > 0$ and $\rho' > \rho_1$ such that

$$(4.3) \quad \mathbf{P} \left(\limsup_{n \rightarrow \infty} Z_n > \rho'_1 \right) \geq \epsilon > 0.$$

We prove this by contradiction using a comparison argument with a RRU model. The proof involves last exit time arguments. Now, suppose (4.3) holds and let $A_1 := \{\limsup_{n \rightarrow \infty} Z_n > \rho'_1\}$. Let

$$R_1 := \left\{ k \geq 0 : \hat{\rho}_{1,k} \geq \frac{\rho'_1 + \rho_1}{2} \right\},$$

and denote the last time the process $\{\hat{\rho}_{1,n}; n \geq 1\}$ is above $(\rho'_1 + \rho_1)/2$ by

$$t_{\frac{\rho'_1 + \rho_1}{2}} = \begin{cases} \sup\{R_1\} & \text{if } R_1 \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\hat{\rho}_{1,n} \xrightarrow{a.s.} \rho_1$, then we have that $\mathbf{P} \left(t_{\frac{\rho'_1 + \rho_1}{2}} < \infty \right) = 1$. Hence, there exists $n_\epsilon \in \mathbb{N}$ such that

$$(4.4) \quad \mathbf{P} \left(t_{\frac{\rho'_1 + \rho_1}{2}} > n_\epsilon \right) \leq \frac{\epsilon}{2}.$$

Setting $B_1 := \left\{ t_{\frac{\rho'_1 + \rho_1}{2}} > n_\epsilon \right\}$ and using (4.4), it follows that

$$\epsilon \leq \mathbf{P}(A_1) \leq \epsilon/2 + \mathbf{P}(A_1 \cap B_1^c).$$

Now, we show that $\mathbf{P}(A_1 \cap B_1^c) = 0$. Setting

$$C_1 = \left\{ \omega \in \Omega : \liminf_{n \rightarrow \infty} Z_n < \frac{\rho'_1 + \rho_1}{2} \right\},$$

we decompose $\mathbf{P}(A_1 \cap B_1^c)$ as follows:

$$\mathbf{P}(A_1 \cap B_1^c) \leq \mathbf{P}(E_1) + \mathbf{P}(E_2),$$

where $E_1 = A_1 \cap B_1^c \cap C_1$ and $E_2 = A_1 \cap B_1^c \cap C_1^c$.

Consider the term $\mathbf{P}(E_2)$. Note that on the set C_1^c , we have $\left\{ \liminf_{n \rightarrow \infty} Z_n \geq \frac{\rho'_1 + \rho_1}{2} \right\}$ and on the set B_1^c we have $\{\hat{\rho}_{1,n} \leq \frac{\rho'_1 + \rho_1}{2}\}$ for any $n \geq n_\epsilon$. Hence, since $B_1^c \cap C_1^c \supset E_2$, on the set E_2 we have that $W_{1,n} = \mathbf{1}_{\{Z_n \leq \hat{\rho}_{1,n}\}} \xrightarrow{a.s.} 0$. Then, letting $\tau_W := \sup\{k \geq 1 : W_{1,k} = 1\}$ we have $\mathbf{P}(E_2 \cap \{\tau_W < \infty\}) = \mathbf{P}(E_2)$ and, on the set E_2 , for any $n \geq \tau_W$ the ARRU model can be written as follows:

$$\begin{cases} Y_{1,n+1} = Y_{1,\tau_W} \\ Y_{2,n+1} = Y_{2,\tau_W} + \sum_{i=\tau_W}^{n+1} (1 - X_i) D_{2,i}, \end{cases}$$

where $W_{1,i-1} = 0$ for any $i \geq \tau_W$, and $W_{2,i-1} = 1$ because $W_{2,i-1} + W_{2,i-1} \geq 1$ by construction. Now, consider an RRU model $\{Z_i^R; i \geq 1\}$ with initial composition $(Y_{1,0}^R, Y_{2,0}^R) = (Y_{1,\tau_W}, Y_{2,\tau_W})$ a.s.; the reinforcements are defined as $D_{1,i}^R = 0$ and $D_{2,i}^R = D_{2,\tau_W+i}$ for any $i \geq 1$ a.s.; the drawing process is modeled by $X_{i+1}^R := \mathbf{1}_{\{U_i^R < Z_i^R\}}$ and $U_i^R = U_{\tau_W+i}$ a.s., where $\{U_n; n \geq 1\}$ is the sequence such that $X_{n+1} = \mathbf{1}_{\{U_n < Z_n\}}$ for any $n \geq 1$. Formally, this RRU model can be described for any $n \geq 1$ as follows:

$$\begin{cases} Y_{1,n+1}^R = Y_{1,0}^R = Y_{1,\tau_W} \\ Y_{2,n+1}^R = Y_{2,0}^R + \sum_{i=0}^{n+1} (1 - X_i^R) D_{2,i}^R = Y_{2,\tau_W} + \sum_{i=\tau_W}^{n+\tau_W+1} (1 - X_i) D_{2,i}. \end{cases}$$

Hence, on the set E_2 we have that

$$(Y_{1,n}, Y_{2,n}) = (Y_{1,n-\tau_W}^R, Y_{2,n-\tau_W}^R) \quad a.s.,$$

for any $n \geq \tau_W$. Since from [18] $\mathbf{P}(\limsup_{n \rightarrow \infty} Z_n^R = 0) = 1$, on the set E_2 we have that $\{\limsup_{n \rightarrow \infty} Z_n = 0\}$. This is incompatible with the set A_1 which includes E_2 . Hence $\mathbf{P}(E_2) = 0$.

We now turn to the proof that $\mathbf{P}(E_1) = 0$. To this end, let

$$\tau_\epsilon := \inf \left\{ k \geq n_\epsilon : \left\{ Z_k < \frac{\rho'_1 + \rho_1}{2} \right\} \cap \left\{ Y_k > \frac{b}{(\rho'_1 - \rho_1)/2} \right\} \right\}$$

and note that, since by Lemma 4.2 $Y_n \xrightarrow{a.s.} \infty$, $\mathbf{P}(C_1 \cap \{\tau_\epsilon < \infty\}) = \mathbf{P}(C_1)$. Moreover, on the set B_1^c we have that $\{\hat{\rho}_{1,n} \leq \frac{\rho'_1 + \rho_1}{2}\}$ for any $n \geq n_\epsilon$. We now show by induction that on the set $B_1^c \cap C_1$ we have $\{Z_n < \rho'_1 \ \forall n \geq \tau_\epsilon\}$. By definition we have $Z_{\tau_\epsilon} < \frac{\rho'_1 + \rho_1}{2}$, and by Lemma 3.1 this implies $Z_{\tau_\epsilon+1} < \rho'_1$; now, consider an arbitrary $n > \tau_\epsilon$; if $Z_n < \frac{\rho'_1 + \rho_1}{2}$, then by Lemma 3.1 we have $Z_{n+1} < \rho'_1$; if $\frac{\rho'_1 + \rho_1}{2} < Z_n < \rho'_1$ we have $W_{1,n} = 0$ and so $Z_{n+1} \leq Z_n < \rho'_1$. Hence, since $B_1^c \cap C_1 \subset E_1$, on the set E_1 we have $\{Z_n < \rho'_1 \ \forall n \geq \tau_\epsilon\}$. This is incompatible with the set A_1 which also includes E_1 . Hence $\mathbf{P}(E_1) = 0$.

Combining all together we have $\epsilon \leq \epsilon/2 + \mathbf{P}(E_1) + \mathbf{P}(E_2) = \epsilon/2$, which is impossible. Thus, we conclude that $\mathbf{P}(A_1^c) = \mathbf{P}(\limsup_{n \rightarrow \infty} Z_n \leq \rho_1) = 1$.

We now prove that $\mathbf{P}(\limsup_{n \rightarrow \infty} Z_n \geq \rho_1) = 1$. To this end, we now show that there cannot exist $\epsilon > 0$ and $\rho' < \rho_1$ such that

$$(4.5) \quad \mathbf{P}\left(\limsup_{n \rightarrow \infty} Z_n < \rho'_1\right) \geq \epsilon > 0.$$

We prove this by contradiction, using a comparison argument with a RRU model. Now suppose (4.5) holds and let $A_2 := \{\limsup_{n \rightarrow \infty} Z_n < \rho'_1\}$.

Let

$$R_2 := \left\{ k \geq 0 : \hat{\rho}_{1,k} < \frac{\rho'_1 + \rho_1}{2} \right\},$$

and define the last time the process $\{\hat{\rho}_{1,n}; n \geq 1\}$ is less than $(\rho'_1 + \rho_1)/2$ by

$$\tau_{\frac{\rho'_1 + \rho_1}{2}} = \begin{cases} \sup\{R_2\} & \text{if } R_2 \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\hat{\rho}_{1,n} \xrightarrow{a.s.} \rho_1$, then we have that $\mathbf{P}\left(\tau_{\frac{\rho'_1 + \rho_1}{2}} < \infty\right) = 1$. Hence, there exists $n_\epsilon \in \mathbb{N}$ such that

$$(4.6) \quad \mathbf{P}\left(\tau_{\frac{\rho'_1 + \rho_1}{2}} > n_\epsilon\right) \leq \frac{\epsilon}{2}.$$

Setting $B_2 := \left\{ \tau_{\frac{\rho'_1 + \rho_1}{2}} > n_\epsilon \right\}$ and using (4.6), it follows that

$$\epsilon \leq \mathbf{P}(A_2) \leq \epsilon/2 + \mathbf{P}(A_2 \cap B_2^c).$$

Let $E_3 := A_2 \cap B_2^c$. We now show that $\mathbf{P}(E_3) = 0$. On the set A_2 , we have $\{\liminf_{n \rightarrow \infty} Z_n \leq \rho'_1\}$ and on the set B_2^c , we have $\{\hat{\rho}_{1,n} \geq \frac{\rho'_1 + \rho_1}{2}\}$ for any $n \geq$

n_ϵ . Hence, on the set E_3 we have that $W_{1,n} = \mathbf{1}_{\{Z_n \leq \hat{\rho}_{1,n}\}} \xrightarrow{a.s.} 1$. Then, letting $\tau_W := \sup\{k \geq 1 : W_{1,k} = 0\}$ we have $\mathbf{P}(E_3 \cap \{\tau_W < \infty\}) = \mathbf{P}(E_3)$. Now, analogously to the proof of $\mathbf{P}(E_2) = 0$, we can use comparison arguments with the RRU model to show that on the set E_3 we have $\{\limsup_{n \rightarrow \infty} Z_n = 1\}$. This is incompatible with the set A_2 , which also includes E_3 . Hence $\mathbf{P}(E_3) = 0$.

Combining all together we have $\epsilon \leq \epsilon/2 + \mathbf{P}(E_3) = \epsilon/2$, which is impossible. Thus, we conclude that the event $A_2^c = \{\limsup_{n \rightarrow \infty} Z_n \geq \rho_1\}$ occurs with probability one.

Proof of part (b):

In part (a), we have shown that $\mathbf{P}(\limsup_{n \rightarrow \infty} Z_n = \rho_1) = 1$. Therefore, if the process $\{Z_n; n \geq 1\}$ converges almost surely, then its limit has to be equal to ρ_1 . First, let d, u, γ and ρ'_1 ($d < u < \gamma < \rho'_1 < \rho_1$) be four constants in $(0, 1)$. Let $\{\tau_j(d, u); j \geq 1\}$ and $\{t_j(d, u); j \geq 1\}$ be the sequences of random variables defined in (4.2). Since d and u are fixed in this proof, we sometimes denote $\tau_j(d, u)$ by τ_j and $t_j(d, u)$ by t_j . It is easy to see that τ_n and t_n are stopping times with respect to $\{\mathcal{F}_n; n \geq 1\}$.

Recall that, by Lemma 4.3, we have that for every $0 < d < u < 1$

$$\begin{aligned} Z_n \text{ converges a.s.} &\Leftrightarrow \mathbf{P}(t_n(d, u) < \infty) \rightarrow 0, \\ &\Leftrightarrow \sum_{n=1}^{\infty} \mathbf{P}(t_{n+1}(d, u) = \infty | t_n(d, u) < \infty) = \infty. \end{aligned}$$

Now, to prove that Z_n converges a.s., it is sufficient to show that

$$\mathbf{P}(t_n(d, u) < \infty) \rightarrow 0,$$

for all $0 < d < u < 1$. Suppose Z_n does not converges a.s.. This implies that $\mathbf{P}(t_n < \infty) \downarrow \phi_1 > 0$, since $\mathbf{P}(t_n < \infty)$ is a non-increasing sequence. We will show that for large j there exists a constant $\phi < 1$ dependent on ϕ_1 , such that

$$(4.7) \quad \mathbf{P}(t_{j+1} < \infty | t_j < \infty) \leq \phi.$$

This result implies that $\sum_n \mathbf{P}(t_{n+1} = \infty | t_n < \infty) = \infty$, establishing by Lemma 4.3 that $\mathbf{P}(t_n < \infty) \rightarrow 0$, which is a contradiction.

Consider the term $\mathbf{P}(t_{i+1} < \infty | t_i < \infty)$. First, let us denote by $\tau_{\rho'_1}$ the last time the process $\hat{\rho}_{1,n}$ is below ρ'_1 , i.e.

$$\tau_{\rho'_1} = \begin{cases} \sup\{n \geq 1 : \hat{\rho}_{1,n} \leq \rho'_1\} & \text{if } \{n \geq 1 : \hat{\rho}_{1,n} \leq \rho'_1\} \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\hat{\rho}_{1,n} \xrightarrow{a.s.} \rho_1$, we have that $\mathbf{P}(\tau_{\rho'_1} < \infty) = 1$. Hence, for any $\epsilon \in (0, \frac{1}{2})$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$(4.8) \quad \frac{1}{\phi_1} \mathbf{P}(\tau_{\rho'_1} > n_\epsilon) \leq \epsilon.$$

By denoting $\mathbf{P}_i(\cdot) = \mathbf{P}(\cdot | t_i < \infty)$ and using $t_i \leq \tau_i \leq t_{i+1}$ we obtain

$$\mathbf{P}(t_{i+1} < \infty | t_i < \infty) \leq \mathbf{P}_i(\tau_i < \infty).$$

Hence

$$(4.9) \quad \mathbf{P}_i(\tau_i < \infty) \leq \mathbf{P}_i(\{\tau_i < \infty\} \cap \{\tau_{\rho'_1} \leq n_\epsilon\}) + \mathbf{P}_i(\tau_{\rho'_1} > n_\epsilon).$$

We start with the second term in (4.9). Note that

$$\mathbf{P}_i(\tau_{\rho'_1} > n_\epsilon) \leq \frac{\mathbf{P}(\tau_{\rho'_1} > n_\epsilon)}{\mathbf{P}(t_i < \infty)} \leq \frac{\mathbf{P}(\tau_{\rho'_1} > n_\epsilon)}{\phi_1} \leq \epsilon,$$

where the last inequality follows from (4.8).

Now, consider the first term in (4.9). Since the probability is conditioned to the set $\{t_i < \infty\}$, in what follows we will consider the urn process at times n after the stopping time t_i . Since we want to show (4.7) for large i , we can choose an integer $i \geq n_\epsilon$ and

$$i > \log_{\frac{u(1-d)}{d(1-u)}} \left(\frac{b}{Y_0(\gamma - u)} \right),$$

so that

- (i) $t_i \geq i \geq n_\epsilon$ a.s.;
- (ii) from Lemma 4.4, we have that $Y_{\tau_i} > b/(\gamma - u)$ a.s.

These two properties imply respectively that, on the set $\{n \geq t_i\}$

- (i) $\hat{\rho}_{1,n} \geq \rho'_1$, since from $\{\tau_{\rho'_1} \leq n_\epsilon\}$ we have that $n \geq \tau_{\rho'_1}$;
- (ii) $Z_{t_i} \in (u, \gamma)$, since $Z_{t_i-1} \leq u$ and $Z_{t_i} > u$ and from Lemma 3.1 we have that $|Z_n - Z_{n-1}| < (\gamma - u)$.

Now, let us define two sequences of stopping times $\{t_n^*; n \geq 1\}$ and $\{\tau_n^*; n \geq 1\}$, where t_n^* represents the first time after τ_{n-1}^* the process Z_{t_i+n} up-crosses ρ'_1 , while τ_n^* represents the first time after t_n^* the process Z_{t_i+n}

down-crosses γ . Formally, let $\tau_0^* = 0$ and define for every $j \geq 1$ two stopping times

$$(4.10) \quad \begin{aligned} t_j^* &= \begin{cases} \inf\{n > \tau_{j-1}^* : Z_{\tau_i+n} > \rho'_1\} & \text{if } \{n > \tau_j^* : Z_{\tau_i+n} > \rho'_1\} \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases} \\ \tau_j^* &= \begin{cases} \inf\{n > t_j^* : Z_{\tau_i+n} \leq \gamma\} & \text{if } \{n > t_{j-1}^* : Z_{\tau_i+n} \leq \gamma\} \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Note that, since $Z_{t_i+\tau_{j-1}^*} \geq \gamma$ and $Z_{t_i+\tau_j^*} < \gamma$, from (ii) we have that $Z_{t_i+\tau_j^*} \in (u, \gamma)$.

For any $j \geq 0$, let $\{\tilde{Z}_n^j; n \geq 1\}$ be a RRU model defined as follows:

- (1) $(\tilde{Y}_{1,0}^j, \tilde{Y}_{2,0}^j) = (Y_{1,t_i+\tau_j^*}, Y_{1,t_i+\tau_j^*} \frac{u+d}{2-u-d})$ a.s., which implies that $\tilde{Z}_0^j = \frac{u+d}{2}$;
- (2) the drawing process is modeled by $\tilde{X}_{n+1}^j = \mathbf{1}_{\{\tilde{U}_{n+1}^j < \tilde{Z}_n^j\}}$, where $\tilde{U}_{n+1}^j = U_{t_i+\tau_j^*+n+1}$ a.s. and U_n is such that $X_n = \mathbf{1}_{\{U_n < Z_{n-1}\}}$;
- (3) the reinforcements are defined as $\tilde{D}_{2,n+1}^j = D_{2,t_i+\tau_j^*+n+1} + (m_1 - m_2)$, $\tilde{D}_{1,n+1}^j = D_{1,t_i+\tau_j^*+n+1}$ a.s.; this means $\mathbf{E}[\tilde{D}_{1,n}^j] = \mathbf{E}[\tilde{D}_{2,n}^j]$ for any $n \geq 1$;
- (4) the urn process evolves as a RRU model, i.e. for any $n \geq 0$

$$\begin{cases} \tilde{Y}_{1,n+1}^j = \tilde{Y}_{1,n}^j + \tilde{X}_{n+1}^j \tilde{D}_{1,n+1}^j, \\ \tilde{Y}_{2,n+1}^j = \tilde{Y}_{2,n}^j + (1 - \tilde{X}_{n+1}^j) \tilde{D}_{2,n+1}^j, \\ \tilde{Y}_{n+1}^j = \tilde{Y}_{1,n+1}^j + \tilde{Y}_{2,n+1}^j, \\ \tilde{Z}_{n+1}^j = \frac{\tilde{Y}_{1,n+1}^j}{\tilde{Y}_{n+1}^j}. \end{cases}$$

We will compare the process $\{Z_{t_i+n}; n \geq 1\}$ with the ARR model process $\{\tilde{Z}_n^j; n \geq 1\}$. Note that at time n , we have defined only the processes \tilde{Z}_n^j such that $\tau_j^* < n$.

We will prove, by induction, that on the set $\{\tau_{\rho'_1} \leq n_\epsilon\}$, for any $j \in \mathbb{N}$ and for any $n \leq t_{j+1}^* - \tau_j^*$

$$(4.11) \quad \tilde{Z}_n^j < Z_{t_i+\tau_j^*+n}, \quad \tilde{Y}_{2,n}^j \geq Y_{2,t_i+\tau_j^*+n}, \quad \tilde{Y}_{1,n}^j < Y_{1,t_i+\tau_j^*+n}.$$

In other words, we will show, provided that $t_i > \tau_{\rho'_1}$, that for each $j \geq 1$ the process \tilde{Z}_n^j is always dominated by the original process $Z_{t_i+\tau_j^*+n}$, as long as

$Z_{t_i+\tau_j^*+n}$ is dominated by ρ'_1 (i.e. for $n \leq t_{j+1}^* - \tau_j^*$). By construction we have that

$$\tilde{Z}_0^j = \frac{d+u}{2} < u < Z_{t_i+\tau_j^*}, \quad \tilde{Y}_{1,0}^j = Y_{1,t_i+\tau_j^*}$$

which immediately implies $\tilde{Y}_{2,0}^j > Y_{2,t_i+\tau_j^*}$. To this end, we assume (4.11) by induction hypothesis. First, we will show that $\tilde{Y}_{2,n+1}^j > Y_{2,t_i+\tau_j^*+n+1}$. Since from (4.11) $\tilde{Z}_n^j < Z_{t_i+\tau_j^*+n}$ for $n \leq t_{j+1}^* - \tau_j^*$, by construction we obtain that

$$\tilde{X}_{n+1}^j = \mathbf{1}_{\{\tilde{U}_n^j < \tilde{Z}_n^j\}} \leq \mathbf{1}_{\{U_{t_i+\tau_j^*+n} < Z_{t_i+\tau_j^*+n}\}} = X_{t_i+\tau_j^*+n+1}.$$

As a consequence, since $W_n \leq 1$ for any $n \geq 1$, we have that

$$\begin{aligned} \left(Y_{2,t_i+\tau_j^*+n+1} - Y_{2,t_i+\tau_j^*+n} \right) &= \left(1 - X_{t_i+\tau_j^*+n+1} \right) D_{2,t_i+\tau_j^*+n+1} W_{2,t_i+\tau_j^*+n} \\ &\leq (1 - \tilde{X}_{n+1}^j) \tilde{D}_{2,n+1}^j \\ &= \left(\tilde{Y}_{2,n+1}^j - \tilde{Y}_{2,n}^j \right), \end{aligned}$$

which using hypothesis (4.11) implies $\tilde{Y}_{2,n+1}^j > Y_{2,t_i+\tau_j^*+n+1}$. Similarly, we now show that $\tilde{Y}_{1,n+1}^j \leq Y_{1,t_i+\tau_j^*+n+1}$. We have

$$\left(Y_{1,t_i+\tau_j^*+n+1} - Y_{1,t_i+\tau_j^*+n} \right) = X_{t_i+\tau_j^*+n+1} D_{1,t_i+\tau_j^*+n+1} W_{1,t_i+\tau_j^*+n}.$$

From (i) we have that, as long as Z remains below ρ'_1 , Z is also above the process $\hat{\rho}_{1,n}$. Since we consider the behavior of $Z_{t_i+\tau_j^*+n}$ when it is below ρ'_1 , i.e. $n \leq \tau_{j+1}^* - t_j^*$, we have that $W_{1,t_i+\tau_j^*+n} = 1$. Thus,

$$\left(Y_{1,t_i+\tau_j^*+n+1} - Y_{1,t_i+\tau_j^*+n} \right) \geq \tilde{X}_{n+1}^j \tilde{D}_{1,n+1}^j = \left(\tilde{Y}_{1,n+1}^j - \tilde{Y}_{1,n}^j \right),$$

which using hypothesis (4.11) implies $\tilde{Y}_{1,n+1}^j \leq Y_{1,t_i+\tau_j^*+n+1}$. Thus, we have shown that, on the set $\{\tau_{\rho'_1} \leq n_\epsilon\}$, for any $n \leq t_{j+1}^* - \tau_j^*$, $\tilde{Z}_{n+1}^j < Z_{t_i+\tau_j^*+n+1}$, $\tilde{Y}_{1,n+1}^j \leq Y_{1,t_i+\tau_j^*+n+1}$ and $\tilde{Y}_{2,n+1}^j > Y_{2,t_i+\tau_j^*+n+1}$ hold.

Now, for any $j \geq 1$, let T_j be the stopping time for \tilde{Z}_n^j to exit from (d, u) , i.e.:

$$T_j = \begin{cases} \inf\{R_3\} & \text{if } R_3 \neq \emptyset; \\ +\infty & \text{otherwise,} \end{cases}$$

where $R_3 := \{n \geq 1 : \tilde{Z}_n^j \leq d \text{ or } \tilde{Z}_n^j \geq u\}$. Note that, on the set $\{\tau_{\rho'_1} \leq n_\epsilon\}$,

$$\begin{aligned} \{\tau_i < \infty\} &= \left\{ \inf_{n \geq 1} \{Z_{t_i+n}\} < d \right\} \subset \left\{ \bigcup_{j: \tau_j^* \leq n} \left\{ \inf_{n \geq 1} \{\tilde{Z}_{n-\tau_j^*}^j\} < d \right\} \right\} \\ &\subset \left\{ \bigcup_{j=0}^\infty \{T_j < \infty\} \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{P}_i \left(\{\tau_i < \infty\} \cap \{\tau_{\rho'_1} \leq n_\epsilon\} \right) &\leq \mathbf{P}_i \left(\left\{ \bigcup_{j=0}^\infty \{T_j < \infty\} \right\} \cap \{\tau_{\rho'_1} \leq n_\epsilon\} \right) \\ &\leq \sum_{j=0}^\infty \mathbf{P}_i \left(\{T_j < \infty\} \cap \{\tau_{\rho'_1} \leq n_\epsilon\} \right). \end{aligned}$$

Consider a single term of the series; by setting $h = \frac{u-d}{2}$ we get

$$\begin{aligned} \mathbf{P}_i \left(\{T_j < \infty\} \cap \{\tau_{\rho'_1} \leq n_\epsilon\} \right) &\leq \mathbf{P}_i \left(\left\{ \sup_{n \geq 1} |\tilde{Z}_n^j - \tilde{Z}_0^j| \geq h \right\} \cap \{\tau_{\rho'_1} \leq n_\epsilon\} \right) \\ &\leq \mathbf{P}_i \left(\sup_{n \geq 1} |\tilde{Z}_n^j - \tilde{Z}_0^j| \geq h \right). \end{aligned}$$

Note that $\{\tilde{Z}_n^j; n \geq 1\}$ is the proportion of red balls in a RRU model with same reinforcement means. Then, using Lemma 4.5 we obtain

$$\begin{aligned} \mathbf{P}_i \left(\sup_{n \geq 1} |\tilde{Z}_n^j - \tilde{Z}_0^j| \geq h \right) &= \mathbf{E}_i \left[\mathbf{P} \left(\left\{ \sup_{n \geq 1} |\tilde{Z}_n^j - \tilde{Z}_0^j| \geq h \right\} \middle| \mathcal{F}_{\tau_i+t_j^*} \right) \right] \\ &\leq \mathbf{E}_i \left[\frac{b}{Y_{t_j^*}} \right] \left(\frac{4}{h^2} + \frac{2}{h} \right). \end{aligned}$$

where $\mathbf{E}_i[\cdot] = \mathbf{E}[\cdot | t_i < \infty]$. Moreover, using Lemma 4.4, the right hand side can be expressed as

$$\mathbf{E}_i \left[\frac{b}{Y_{t_i}} \right] \left(\frac{\rho'_1(1-\gamma)}{\gamma(1-\rho'_1)} \right)^j \left(\frac{4}{h^2} + \frac{2}{h} \right).$$

Since from Lemma 4.2 Y_n converges a.s. to infinity, and since $\tau_i \rightarrow \infty$ a.s. because $\tau_i \geq i$, we have that $\mathbf{E}_i[Y_{t_i}^{-1}]$ tends to zero as i increases. As a consequence, we can choose an integer i large enough such that

$$\mathbf{E}_i \left[\frac{b}{Y_{t_i}} \right] \left(\frac{4}{h^2} + \frac{2}{h} \right) \left(\frac{1-\rho'_1}{1-\rho'_1/\gamma} \right) < \frac{1}{2},$$

which setting $\phi = 1/2 + \epsilon$ implies (4.7), i.e.

$$\mathbf{P}(t_{i+1} < \infty | t_i < \infty) \leq \phi < 1.$$

This concludes the proof. \square

PROOF OF COROLLARY 2.1. This corollary has been proved in Proposition 2.1 of [13] for the MRRU. That proof is only based on the fact that the urn proportion Z_n converges a.s. to a value within the interval $(0, 1)$, while the reinforcement rules do not play any role. Hence, the proof used in [13] can be applied to the ARRU, since $Z_n \xrightarrow{a.s.} \rho \in (0, 1)$ for ARRU using Theorem 2.1. \square

4.2. *Proof of the central limit theorem.* Before the proof of Theorem 2.2, we recall that $\{\tau_j; j \geq 1\}$ is the sequence defined in (3.8) as follows:

$$\tau_j := \begin{cases} \inf \left\{ k \geq 1 : \tilde{T}_{j,k} \in [-b, 0] \right\} & \text{if } \left\{ k \geq 1 : \tilde{T}_{j,k} \in [-b, 0] \right\} \neq \emptyset; \\ \infty & \text{otherwise.} \end{cases}$$

Fix $\nu \in (0, 1/2)$ and, for any $j \geq 1$, let $r_j := q^{j \frac{1+\nu}{2}}$ and $\mathcal{R}_j := \{\tau_j > r_j\}$. The following theorem is critical to the proof of Theorem 2.2.

THEOREM 4.1. *Let $\tilde{\rho}_{1,n}$ and $\tilde{\rho}_{2,n}$ be as in (2.5). Then, under assumption (2.1) and (2.7), we have that*

$$(4.12) \quad \mathbf{P}(\mathcal{R}_j, i.o.) = 0.$$

We delay the proof of this theorem to Subsection 4.2.1.

PROOF OF THEOREM 2.2. Wlog assume $m_1 > m_2$, which implies $\rho = \rho_1$. To prove the main result, we establish

- (a) $\sqrt{n} \left(\frac{N_{1,n}}{n} - \frac{\sum_{i=1}^n Z_{i-1}}{n} \right) \xrightarrow{d} \mathcal{N}(0, \rho_1(1 - \rho_1))$, and
- (b) $\sqrt{n} \left(\frac{\sum_{i=1}^n Z_{i-1}}{n} - \frac{\sum_{i=1}^n \tilde{\rho}_{1,i-1}}{n} \right) \xrightarrow{a.s.} 0$.

Finally, result (2.8) is obtained by using Slutsky's Theorem to combine (a) and (b) together.

Proof of part (a): Let us define a random variable $J_{ni} := \frac{1}{\sqrt{n}} (X_i - \mathbf{E}[X_i | \mathcal{F}_{i-1}])$, for any $n, i \in \mathbb{N}$ with $i \leq n$. Then, for each $n \in \mathbb{N}$, the sequence $\{S_{nj} = \sum_{i=1}^j J_{ni}; 1 \leq j \leq n\}$ is a martingale. Now we apply the Martingale CLT (MCLT). First note that $J_{ni}^2 \leq 1/n$ for any $n \in \mathbb{N}$ and $|J_{ni}| < \epsilon$ for any $n \geq \epsilon^{-2}$; thus

$$\sum_{i=1}^n \mathbf{E} \left[J_{ni}^2 \mathbf{1}_{\{|J_{ni}| > \epsilon\}} \mid \mathcal{F}_{i-1} \right] \leq \sum_{i=1}^{\lceil \epsilon^{-2} \rceil + 1} 1/n = \frac{\lceil \epsilon^{-2} \rceil + 1}{n} \rightarrow 0.$$

Also,

$$\begin{aligned} \mathbf{E} [J_{ni}^2 | \mathcal{F}_{i-1}] &= \frac{1}{n} \cdot \mathbf{E} [(X_{ni} - \mathbf{E} [X_{ni} | \mathcal{F}_{i-1}])^2 | \mathcal{F}_{i-1}] \\ &= \frac{Z_{i-1} (1 - Z_{i-1})}{n}, \end{aligned}$$

since $\hat{\rho}_{1,n} \xrightarrow{a.s.} \rho_1$, from Theorem 2.1 we get $Z_n \xrightarrow{a.s.} \rho_1$, which implies

$$\sum_{i=1}^n \mathbf{E} [J_{ni}^2 | \mathcal{F}_{i-1}] = \frac{\sum_{i=1}^n Z_{i-1} (1 - Z_{i-1})}{n} \xrightarrow{a.s.} \rho_1 (1 - \rho_1).$$

From MCLT [14], it follows that

$$\begin{aligned} \frac{1}{\sqrt{n}} \cdot \sum_{i=1}^n (X_i - \mathbf{E} [X_i | \mathcal{F}_{i-1}]) &= \sqrt{n} \left(\frac{\sum_{i=1}^n X_i}{n} - \frac{\sum_{i=1}^n Z_{i-1}}{n} \right) \\ &\xrightarrow{d} \mathcal{N}(0, \rho_1 (1 - \rho_1)). \end{aligned}$$

We now turn to the proof of part (b). We first express

$$\begin{aligned} \sqrt{n} \left(\frac{\sum_{i=1}^n Z_{i-1}}{n} - \frac{\sum_{i=1}^n \tilde{\rho}_{1,i-1}}{n} \right) &= \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} (Z_i - \tilde{\rho}_{1,i}) \\ &= B_{1n} + B_{2n}, \end{aligned}$$

where

$$\begin{aligned} B_{1n} &:= \frac{1}{\sqrt{n}} \sum_{i=0}^{[q^{k_n}]} (Z_i - \tilde{\rho}_{1,i}), \\ B_{2n} &:= \frac{1}{\sqrt{n}} \sum_{i=[q^{k_n}]+1}^{n-1} (Z_i - \tilde{\rho}_{1,i}), \end{aligned}$$

and we recall k_n is defined in (2.6) as $k_n := [\log_q(n)]$. We begin with B_{1n} . Note that

$$\sum_{i=0}^{[q^{k_n}]} (Z_i - \tilde{\rho}_{1,i}) = \sum_{j=1}^{k_n-1} \sum_{i=1}^{d_j} (Z_{q^j+i} - \hat{\rho}_{1,q^j}) = \sum_{j=1}^{k_n-1} \sum_{i=1}^{d_j} (-\Delta_{j,i}),$$

where we recall that $d_j = q^{j+1} - q^j$ and $\Delta_{j,i} = \hat{\rho}_{1,q^j} - Z_{q^j+i}$ for any $j \geq 1$ and $1 \leq i \leq d_j$. Hence

$$|B_{1n}| = \frac{1}{\sqrt{n}} \cdot \left| \sum_{j=1}^{k_n-1} \sum_{i=1}^{d_j} (-\Delta_{j,i}) \right| \leq \frac{1}{\sqrt{n}} \cdot \sum_{j=1}^{k_n-1} \left(\frac{\sum_{i=1}^{d_j} |\Delta_{j,i}|}{\sqrt{d_j}} \right) \sqrt{d_j};$$

similarly

$$|B_{2n}| \leq \frac{1}{\sqrt{n}} \cdot \left(\frac{\sum_{i=1}^{d_{k_n}} |\Delta_{k_n, i}|}{\sqrt{d_{k_n}}} \right) \sqrt{d_{k_n}}.$$

Now, for any $j \geq 1$ define

$$(4.13) \quad b_j := \frac{\sum_{i=1}^{d_j} |\Delta_{k_n, i}|}{\sqrt{d_j}},$$

it follows that

$$|B_{1n}| + |B_{2n}| \leq \frac{1}{\sqrt{n}} \cdot \sum_{j=1}^{k_n} b_j \sqrt{d_j}.$$

Now, we have

$$\begin{aligned} |B_{1n}| + |B_{2n}| &\leq \frac{1}{\sqrt{n}} \cdot \sum_{j=1}^{k_n/2-1} b_j \sqrt{d_j} + \frac{1}{\sqrt{n}} \cdot \sum_{j=k_n/2}^{k_n} b_j \sqrt{d_j} \\ &\leq \left(\frac{\sup_{i \geq 1} \{b_i\}}{\sqrt[4]{n}} \right) \cdot H_{1n} + \left(\sup_{i \geq k_n/2} \{b_i\} \right) \cdot H_{2n}. \end{aligned}$$

where

$$H_{1n} := \frac{1}{\sqrt[4]{n}} \sum_{j=1}^{k_n/2-1} \sqrt{d_j}, \quad H_{2n} := \frac{1}{\sqrt{n}} \sum_{j=1}^{k_n} \sqrt{d_j}.$$

Using $d_j = (q-1)q^j$ we express

$$\begin{aligned} H_{1n} &= \frac{\sqrt{q-1}}{\sqrt[4]{n}} \cdot \sum_{j=1}^{k_n/2-1} (\sqrt{q})^j = \left(\frac{\sqrt{q}^{k_n/2} - 1}{\sqrt[4]{n}} \right) \cdot \left(\frac{\sqrt{q-1}}{\sqrt{q}-1} \right), \\ H_{2n} &= \frac{\sqrt{q-1}}{\sqrt{n}} \cdot \sum_{j=1}^{k_n} (\sqrt{q})^j = \left(\frac{\sqrt{q}^{k_n+1} - \sqrt{q}^{k_n/2}}{\sqrt{n}} \right) \cdot \left(\frac{\sqrt{q-1}}{\sqrt{q}-1} \right). \end{aligned}$$

Since $n \geq q^{k_n}$, it follows that $H_{1n} \leq C$ and $H_{2n} \leq \sqrt{q}C$, where $C = \left(\frac{\sqrt{q-1}}{\sqrt{q}-1} \right)$. Thus,

$$|B_{1n}| + |B_{2n}| \leq \left(\frac{\sup_{i \geq 1} \{b_i\}}{\sqrt[4]{n}} \right) \cdot C + \left(\sup_{i \geq k_n/2} \{b_i\} \right) \cdot \sqrt{q}C.$$

To conclude the proof we will show that $b_j \xrightarrow{a.s.} 0$.

First, fix an arbitrary constant $\nu \in (0, 1/2)$ and let $r_j := q^{j\frac{1+\nu}{2}}$ for any $j \geq 1$; then, write

$$\begin{aligned} b_j &= \frac{1}{\sqrt{d_j}} \sum_{i=1}^{d_j} |\Delta_{j,i}| = \left(\frac{1}{\sqrt{d_j}} \sum_{i=1}^{r_j} |\Delta_{j,i}| \right) + \left(\frac{1}{\sqrt{d_j}} \sum_{i=r_j+1}^{d_j} |\Delta_{j,i}| \right) \\ &= F_{1j} + F_{2j}, \end{aligned}$$

Let us consider term F_{1j} , we have that

$$F_{1j} = \frac{r_j}{\sqrt{d_j}} \cdot \left(\frac{1}{r_j} \sum_{i=1}^{r_j} |\Delta_{j,i}| \right) = \frac{\left[q^{j\frac{\nu}{2}} \right]}{\sqrt{q-1}} \cdot \left(\frac{1}{r_j} \sum_{i=1}^{r_j} |\Delta_{j,i}| \right),$$

since $d_j = (q-1)q^j$ and $r_j/\sqrt{q^j} = q^{j\frac{\nu}{2}}$. Now, for any $i = 1, \dots, r_j$ we note that

$$|\Delta_{j,i}| \leq |Z_{q^j+i} - Z_{q^j}| + |\Delta_{j-1,d_{j-1}}| + |\hat{\rho}_{1,q^{j-1}} - \hat{\rho}_{1,q^j}|;$$

hence, we have

$$F_{1j} \leq E_{1j} + E_{2j} + E_{3j},$$

where

$$\begin{aligned} E_{1j} &:= \frac{\left[q^{j\frac{\nu}{2}} \right]}{\sqrt{q-1}} \cdot \left(\frac{1}{r_j} \sum_{i=1}^{r_j} |Z_{q^j+i} - Z_{q^j}| \right), \\ E_{2j} &:= \frac{\left[q^{j\frac{\nu}{2}} \right]}{\sqrt{q-1}} \cdot |\Delta_{j-1,d_{j-1}}|, \\ E_{3j} &:= \frac{\left[q^{j\frac{\nu}{2}} \right]}{\sqrt{q-1}} \cdot |\hat{\rho}_{1,q^{j-1}} - \hat{\rho}_{1,q^j}|. \end{aligned}$$

Let us consider the term E_{1j} . Since from Lemma 3.1 we have $|Z_k - Z_{k-1}| \leq b/Y_{k-1}$, we have that

$$E_{1j} \leq \frac{\left[q^{j\frac{\nu}{2}} \right]}{\sqrt{q-1}} \cdot \frac{br_j}{Y_{q^j}} = \left(\frac{b}{\sqrt{q-1}} \right) \cdot \left(\frac{q^{j(\frac{1}{2}+\nu)}}{Y_{q^j}} \right).$$

Then, by using Markov's inequality we obtain

$$\sum_{j=1}^{\infty} \mathbf{P} \left(\frac{q^{j(\frac{1}{2}+\nu)}}{Y_{q^j}} > \epsilon \right) \leq \frac{1}{\epsilon} \sum_{j=1}^{\infty} \mathbf{E} \left[\frac{q^j}{Y_{q^j}} \right] q^{-j(\frac{1}{2}-\nu)} \leq \frac{C}{\epsilon} \sum_{j=1}^{\infty} q^{-j(\frac{1}{2}-\nu)} < \infty,$$

where $C = \sup_{k \geq 1} \{\mathbf{E}[k/Y_k]\}$ is finite from Theorem 3.1. Thus, from the Borel-Cantelli lemma it follows that $E_{1j} \xrightarrow{a.s.} 0$.

Now, consider the term E_{2j} . We have

$$\begin{aligned} \mathbf{P} \left(\lim_{k \rightarrow \infty} \cup_{j \geq k} \{E_{2j} > \epsilon\} \right) &\leq \mathbf{P} \left(\lim_{k \rightarrow \infty} \cup_{j \geq k} \mathcal{R}_j \right) \\ &\quad + \mathbf{P} \left(\lim_{k \rightarrow \infty} \cup_{j \geq k} \left\{ \frac{\left\lfloor q^{(j+1)\frac{\nu}{2}} \right\rfloor}{\sqrt{q-1}} \cdot |\Delta_{j,d_j}| > \epsilon \right\} \cap \mathcal{R}_j^c \right). \end{aligned}$$

where the term $\mathbf{P}(\lim_{k \rightarrow \infty} \cup_{j \geq k} \mathcal{R}_j) = 0$ from Theorem 4.1. Then, by using Markov's inequality we obtain

$$\sum_{j=1}^{\infty} \mathbf{P} \left(\left\{ \frac{\left\lfloor q^{(j+1)\frac{\nu}{2}} \right\rfloor}{\sqrt{q-1}} \cdot |\Delta_{j,d_j}| > \epsilon \right\} \cap \mathcal{R}_j^c \right) \leq M,$$

where

$$M := \frac{1}{\epsilon} \sum_{j=1}^{\infty} \mathbf{E} \left[\frac{\left\lfloor q^{(j+1)\frac{\nu}{2}} \right\rfloor}{\sqrt{q-1}} \cdot |\Delta_{j,d_j}| \mathbf{1}_{\mathcal{R}_j^c} \right].$$

Now, for any $j \geq 1$ let us introduce the set $\mathcal{Q}_j := \{\tau_j > d_j\}$. Using $\mathcal{R}_j^c \subseteq \mathcal{Q}_j^c$ from $r_j \leq d_j$, and by multiplying and dividing by q^{j+1} , we have that

$$\begin{aligned} M &= \frac{1}{\epsilon \sqrt{q-1}} \sum_{j=1}^{\infty} \mathbf{E} \left[q^{j+1} |\Delta_{j,d_j}| \mathbf{1}_{\mathcal{R}_j^c} \right] \cdot q^{-(j+1)(1-\frac{\nu}{2})} \\ &\leq \frac{1}{\epsilon \sqrt{q-1}} \sum_{j=1}^{\infty} \mathbf{E} \left[q^{j+1} |\Delta_{j,d_j}| \mathbf{1}_{\mathcal{Q}_j^c} \right] \cdot q^{-(j+1)(1-\frac{\nu}{2})} \\ &\leq \frac{1}{\epsilon \sqrt{q-1}} \left(\sup_{k \geq 1} \left\{ \mathbf{E} \left[q^{k+1} |\Delta_{k,d_k}| \mathbf{1}_{\mathcal{Q}_k^c} \right] \right\} \right) \sum_{j=1}^{\infty} q^{-(j+1)(1-\frac{\nu}{2})} \\ &< \infty, \end{aligned}$$

using Theorem 3.2 and the result follows from the Borel-Cantelli lemma.

Let us consider the term E_{3j} . For any $\epsilon > 0$, by using Markov's inequality we have

$$\mathbf{P}(E_{3j} > \epsilon) \leq \frac{1}{\epsilon \sqrt{q-1}} \mathbf{E} \left[q^{j\frac{\nu}{2}} \cdot |\hat{\rho}_{1,q^j} - \hat{\rho}_{1,q^{j-1}}| \right].$$

The right-hand side (RHS) of the above expression can be rewritten as

$$\frac{q^{-j(\frac{1-\nu}{2})}}{\epsilon\sqrt{q-1}} \mathbf{E} \left[q^{\frac{j}{2}} \cdot |\hat{\rho}_{1,q^j} - \hat{\rho}_{1,q^{j-1}}| \right],$$

Now, by decomposing the last expectation into

$$\mathbf{E} \left[q^{\frac{j}{2}} \cdot |\hat{\rho}_{1,q^j} - \hat{\rho}_{1,q^{j-1}}| \right] = \mathbf{E} \left[q^{\frac{j}{2}} \cdot |\rho_1 - \hat{\rho}_{1,q^{j-1}}| \right] + \mathbf{E} \left[q^{\frac{j}{2}} \cdot |\rho_1 - \hat{\rho}_{1,q^j}| \right]$$

we can see that

$$\sum_{j=1}^{\infty} \mathbf{P}(E_{3j} > \epsilon) \leq \left(\frac{2 \sup_{k \geq 1} \left\{ \mathbf{E} \left[q^{\frac{k}{2}} \cdot |\rho_1 - \hat{\rho}_{1,q^k}| \right] \right\}}{\epsilon\sqrt{q-1}} \right) \sum_{j=1}^{\infty} q^{-j(\frac{1-\nu}{2})},$$

which is finite because of (2.7). Hence, by another application the Borel-Cantelli lemma, $E_{3j} \xrightarrow{a.s.} 0$; then, we have $F_{1j} \xrightarrow{a.s.} 0$.

Finally, let us consider term F_{2j} . First, we multiply and divide by $(d_j - r_j) q^{-\frac{j}{2}}$ to obtain $F_{2j} = c_j F_{3j}$, where

$$c_j = \frac{d_j - r_j}{q^{\frac{j}{2}} \sqrt{d_j}}, \quad F_{3j} = \frac{1}{d_j - r_j} \sum_{i=r_j+1}^{d_j} q^{\frac{j}{2}} |\Delta_{j,i}|.$$

Since $c_j \rightarrow \sqrt{q-1}$, let us focus on F_{3j} . Since $\mathbf{P}(\mathcal{R}_j, i.o.) = 0$ (Theorem 4.1), it is sufficient to show that $F_{3j} \mathbf{1}_{\mathcal{R}_j^C} \xrightarrow{a.s.} 0$. For any $\epsilon > 0$, by Markov's inequality it follows that

$$\mathbf{P}(\{F_{3j} > \epsilon\} \cap \mathcal{R}_j^C) \leq \frac{1}{\epsilon} \left(\frac{1}{d_j - r_j} \sum_{i=r_j+1}^{d_j} \mathbf{E} \left[q^{\frac{j}{2}} |\Delta_{j,i}| \mathbf{1}_{\mathcal{R}_j^C} \right] \right),$$

Now, since

$$\sum_{i=r_j+1}^{d_j} \mathbf{E} \left[q^{\frac{j}{2}} |\Delta_{j,i}| \mathbf{1}_{\mathcal{R}_j^C} \right] \leq (d_j - r_j) \left(\max_{i=r_j+1, \dots, d_j} \left\{ \mathbf{E} \left[q^{\frac{j}{2}} |\Delta_{j,i}| \mathbf{1}_{\mathcal{R}_j^C} \right] \right\} \right),$$

we have that

$$\begin{aligned} \mathbf{P}(\{F_{3j} > \epsilon\} \cap \mathcal{R}_j^C) &\leq \frac{1}{\epsilon} \left(\max_{i=r_j+1, \dots, d_j} \left\{ \mathbf{E} \left[q^{\frac{j}{2}} |\Delta_{j,i}| \mathbf{1}_{\mathcal{R}_j^C} \right] \right\} \right) \\ &= \frac{1}{\epsilon} \left(\max_{i=r_j+1, \dots, d_j} \left\{ \mathbf{E} \left[q^j |\Delta_{j,i}| \mathbf{1}_{\mathcal{R}_j^C} \right] \right\} \right) q^{-\frac{j}{2}} \\ &\leq \frac{1}{\epsilon} \left(\sup_{k \geq 1} \left\{ \max_{i=[r_k]+1, \dots, d_k} \left\{ \mathbf{E} \left[q^k |\Delta_{k,i}| \mathbf{1}_{\mathcal{R}_k^C} \right] \right\} \right\} \right) q^{-\frac{j}{2}} \\ &\leq C q^{-\frac{j}{2}}, \end{aligned}$$

where the last inequality follows from Theorem 3.2. Now, summing over j we have that

$$\sum_{j=1}^n \mathbf{P}(\{F_{3j} > \epsilon\} \cap \mathcal{R}_j^C) \leq C \sum_{j=1}^n q^{-\frac{j}{2}} < \infty.$$

Now, using the Borel-Cantelli lemma we get that $F_{2j} \xrightarrow{a.s.} 0$, which concludes the proof. \square

4.2.1. Proof of Theorem 4.1.

PROOF. Wlog assume $m_1 > m_2$, which implies $\hat{\rho}_n = \hat{\rho}_{1,n}$ and $\rho = \rho_1$. To prove (4.12) we need to study the sequence of sets $\{\mathcal{R}_j; j \geq 1\}$. On the set \mathcal{R}_j , the urn proportion does not cross the thresholds at times $q^j, \dots, q^j + r_j$. Hence, \mathcal{R}_j will be included in $\mathcal{A}_j \cup \mathcal{B}_j$, where \mathcal{A}_j and \mathcal{B}_j represent the events in which the urn proportion is always above and below, respectively, the thresholds at times $q^j, \dots, q^j + r_j$. To show that \mathcal{A}_j and \mathcal{B}_j cannot occur i.o., we need to appropriately express them by using the following scaling processes:

- (a) $\tilde{T}_{j,k} = Y_{q^j+k} \Delta_{j,k} = Y_{q^j+k} (\hat{\rho}_{1,q^j} - Z_{q^j+k})$, defined for any $j \geq 1$ and any $k = 1, \dots, d_j$. This process models the closeness among the urn proportion and the adaptive threshold.
- (b) $T_n = Y_n (\rho_1 - Z_n)$, defined for any $n \geq 1$. This process models the closeness among the urn proportion and the limit of the threshold's sequence.
- (c) $T_{j,k}^{(\rho_1)} := Y_{q^j+k} (\rho_1 - \hat{\rho}_{1,q^j})$, defined for any $j \geq 1$ and $k = 1, \dots, d_j$. This process models the closeness between the adaptive threshold and its limit.

Let us now define formally the sets \mathcal{A}_j and \mathcal{B}_j . First, note that if the urn proportion crosses the threshold at time $(q^j + k)$, then $\tilde{T}_{q^j+k} \cdot \tilde{T}_{q^j+k-1} < 0$, since only one among \tilde{T}_{q^j+k} and \tilde{T}_{q^j+k-1} is within the interval $[-b, 0]$. Thus, from the definition of τ_j in (3.8), we have that

$$\{\Delta_{j,k-1} \cdot \Delta_{j,k} < 0\} \subseteq \{\tau_j \leq k\}.$$

This implies that

$$\begin{aligned} \mathcal{R}_j &\subset \left\{ \bigcap_{k=1}^{r_j} \{\Delta_{j,k-1} \cdot \Delta_{j,k} > 0\} \right\} \\ &= \left\{ \bigcap_{k=1}^{r_j} \{\Delta_{j,k} < 0\} \right\} \cup \left\{ \bigcap_{k=1}^{r_j} \{\Delta_{j,k} > 0\} \right\}. \end{aligned}$$

Since $Y_{q^j+k}\Delta_{j,k} = T_{q^j+k} - T_{j,k}^{(\rho_1)}$, we can write

$$\mathcal{R}_j \subseteq \mathcal{A}_j \cup \mathcal{B}_j,$$

where

$$\mathcal{A}_j := \cap_{k=1}^{r_j} \mathcal{D}_{j,k}, \quad \mathcal{B}_j := \cap_{k=1}^{r_j} \mathcal{D}_{j,k}^C,$$

and $\mathcal{D}_{j,k} := \left\{ T_{q^j+k} < T_{j,k}^{(\rho_1)} \right\}$ for $k = 1, \dots, r_j$.

The idea to prove that these events cannot occur infinitely often is the following: consider \mathcal{A}_j (for instance) and rewrite the set \mathcal{D}_{j,r_j} as follows:

(4.14)

$$\mathcal{D}_{j,r_j} = \left\{ T_{q^j+r_j} < T_{j,r_j}^{(\rho_1)} \right\} = \left\{ \sum_{i=1}^{r_j} (T_{q^j+i} - T_{q^j+i-1}) < T_{j,r_j}^{(\rho_1)} - T_{q^j} \right\},$$

where the last inequality follows using telescopic series. In the set \mathcal{D}_{j,r_j} we have a sum of bounded random variables, i.e. $(T_{q^j+i} - T_{q^j+i-1})$, whose means are strictly positive on \mathcal{A}_j , because \mathcal{A}_j is included in $\cap_{k=1}^{r_j-1} \mathcal{D}_{j,k}$; hence, provided that the difference $(T_{j,r_j}^{(\rho_1)} - T_{q^j})$ increases with j slower than r_j , we could prove that the set cannot occur infinitely often. Roughly speaking, it means that, if the adaptive threshold $\hat{\rho}_{1,q^j}$ is not far enough from the urn proportion Z_{q^j} , then the average increments of the urn proportion make very likely that Z_{q^j+k} crosses $\hat{\rho}_{1,q^j}$ before $q^j + r_j$. Similar arguments apply for \mathcal{B}_j . More formally, fix $\epsilon > 0$ and define the set \mathcal{C}_j as follows:

$$\mathcal{C}_j := \left\{ |T_{j,r_j}^{(\rho_1)} - T_{q^j}| > \epsilon j^2 q^{\frac{j}{2}} \right\},$$

so that \mathcal{C}_j^C is the set where the difference $|T_{j,r_j}^{(\rho_1)} - T_{q^j}|$ increases with j slower than r_j . Hence, it follows that

$$\mathcal{R}_j \subseteq \{\mathcal{A}_j - \mathcal{C}_j\} \cup \{\mathcal{B}_j - \mathcal{C}_j\} \cup \mathcal{C}_j,$$

and the result (4.12) is obtained by showing that

$$\mathbf{P}(\mathcal{A}_j - \mathcal{C}_j, i.o.) = \mathbf{P}(\mathcal{B}_j - \mathcal{C}_j, i.o.) = \mathbf{P}(\mathcal{C}_j, i.o.) = 0.$$

We will now begin with the proof of $\mathbf{P}(\mathcal{A}_j - \mathcal{C}_j, i.o.) = \mathbf{P}(\mathcal{B}_j - \mathcal{C}_j, i.o.) = 0$. From (4.14) we note that, on the set \mathcal{C}_j^C ,

$$\mathcal{D}_{j,r_j} \subseteq \left\{ \sum_{i=1}^{r_j} (T_{q^j+i} - T_{q^j+i-1}) < \epsilon j^2 q^{\frac{j}{2}} \right\} = \mathcal{E}_j,$$

$$\mathcal{D}_{j,r_j}^C \subseteq \left\{ \sum_{i=1}^{r_j} (T_{q^j+i} - T_{q^j+i-1}) > -\epsilon j^2 q^{\frac{j}{2}} \right\} = \mathcal{F}_j.$$

As a consequence, we have

$$\begin{aligned} \mathcal{A}_j - \mathcal{C}_j &\subseteq \left\{ \cap_{k=1}^{r_j-1} \mathcal{D}_{j,k} \cap \mathcal{E}_j \right\}, \\ \mathcal{B}_j - \mathcal{C}_j &\subseteq \left\{ \cap_{k=1}^{r_j-1} \mathcal{D}_{j,k}^C \cap \mathcal{F}_j \right\}. \end{aligned}$$

Now, consider the increments $(T_{q^j+i} - T_{q^j+i-1})$ for $i = 1, \dots, r_j$ contained in the sets \mathcal{E}_j and \mathcal{F}_j above; recall that

$$\begin{aligned} (T_{q^j+i} - T_{q^j+i-1}) &= \rho_1 (1 - X_{q^j+i}) D_{2,q^j+i} W_{2,q^j+i-1} \\ &\quad - (1 - \rho_1) X_{q^j+i} D_{1,q^j+i} W_{1,q^j+i-1}. \end{aligned}$$

Fix an arbitrarily small $\epsilon_1 > 0$ and introduce two collections of i.i.d. random variables (A_1, \dots, A_{r_j}) and (B_1, \dots, B_{r_j}) defined as follows:

$$\begin{aligned} A_i &:= \rho_1 \left(1 - \mathbf{1}_{\{U_{q^j+i} < \rho_1 + \epsilon_1\}} \right) D_{2,q^j+i}, \\ B_i &:= \rho_1 \left(1 - \mathbf{1}_{\{U_{q^j+i} < \rho_1 - \epsilon_1\}} \right) D_{2,q^j+i} - (1 - \rho_1) \mathbf{1}_{\{U_{q^j+i} < \rho_1 - \epsilon_1\}} D_{1,q^j+i}, \end{aligned}$$

where $(U_{q^j+1}, \dots, U_{q^{j+1}})$ are the i.i.d. $(0, 1)$ uniform random variables such that $X_{q^j+i} := \mathbf{1}_{\{U_{q^j+i} < Z_{q^j+i-1}\}}$.

First note that, by construction, on the set \mathcal{A}_j we have $\cap_{k=1}^{r_j} \{Z_{q^j+k} > \tilde{\rho}_{1,q^j+k}\}$, and hence $\mathcal{A}_j \subset \cap_{k=1}^{r_j} \{W_{1,q^j+k} = 0\}$. Thus, since using (2.7) we have $Z_n \xrightarrow{a.s.} \rho_1$ by Theorem 2.1, on the set \mathcal{A}_j we have that

$$\{(T_{q^j+i} - T_{q^j+i-1}) \geq A_i\}, \quad i = 1, \dots, r_j,$$

occurs with probability 1 as $n \rightarrow \infty$. Similarly, by construction, on the set \mathcal{B}_j we have $\cap_{k=1}^{r_j} \{Z_{q^j+k} < \tilde{\rho}_{1,q^j+k}\}$, and hence $\mathcal{B}_j \subset \cap_{k=1}^{r_j} \{W_{1,q^j+k} = 1\}$. Thus, using $Z_n \xrightarrow{a.s.} \rho_1$, on the set \mathcal{A}_j we have that the event

$$\{(T_{q^j+i} - T_{q^j+i-1}) \leq B_i\}, \quad i = 1, \dots, r_j,$$

occurs with probability 1 as $n \rightarrow \infty$. As a consequence, for large j we have that

$$\begin{aligned} P(\mathcal{A}_j - \mathcal{C}_j, i.o.) &\leq P\left(\sum_{i=1}^{r_j} A_i < \epsilon j^2 q^{\frac{j}{2}}, i.o.\right) \\ P(\mathcal{B}_j - \mathcal{C}_j, i.o.) &\leq P\left(\sum_{i=1}^{r_j} B_i > -\epsilon j^2 q^{\frac{j}{2}}, i.o.\right). \end{aligned}$$

Set

$$P_{Aj} := \mathbf{P} \left(\sum_{i=1}^{r_j} A_i < \epsilon j^2 q^{\frac{j}{2}} \right), \quad \text{and} \quad P_{Bj} := \mathbf{P} \left(\sum_{i=1}^{r_j} B_i > -\epsilon j^2 q^{\frac{j}{2}} \right).$$

We will now use Chernoff's upper bounds on the i.i.d. bounded random variables A_i and B_i (see (3.7)). First notice that

- (1) $\mathbf{E}[A_i] = \rho_1(1 - \rho_1 - \epsilon)m_2 > 0$,
- (2) $\mathbf{E}[B_i] = \rho_1(1 - \rho_1 + \epsilon)m_2 - (1 - \rho_1)(\rho_1 - \epsilon)m_1 < 0$,
- (3) $|A_i|, |B_i| < b$ a.s. for any $i \geq 1$.

Note that P_{Aj} can be written as $\mathbf{P}(S_j \leq c_j \cdot \mathbf{E}[S_j])$, where $S_j = \sum_{i=1}^{r_j} (A_i/b)$ and

$$c_j = \frac{\epsilon j^2 q^{\frac{j}{2}}}{r_j \mathbf{E}[A_1]/b},$$

since $c_j \rightarrow 0$, we can define an integer j_0 such that $c_j < c_0$ for any $j \geq j_0$, so that

$$\mathbf{P}(S_j \leq c_j \cdot \mathbf{E}[S_j]) \leq \mathbf{P}(S_j \leq c_0 \cdot \mathbf{E}[S_j]).$$

Hence, by using (3.7), for any $j \geq j_0$ we have that

$$P_{Aj} \leq \exp \left(-\frac{(1 - c_0)^2}{2} \cdot \mathbf{E}[S_j] \right),$$

which converges to zero exponentially fast since

$$\mathbf{E}[S_j] = r_j \frac{\mathbf{E}[A_1]}{b} \sim q^{j \frac{1+\nu}{2}}.$$

We can repeat the same arguments for P_{Bj} , with the i.i.d. random variables $(-B_i + b)/2b \in (0, 1)$ for $i = 1, \dots, r_j$; in this case, c_j tends to a constant $c < 1$, so that the proof follows with $c_0 \in (c, 1)$. Thus,

$$\sum_{j=1}^{\infty} (P_{Aj} + P_{Bj}) < \infty,$$

yielding

$$\mathbf{P}(\mathcal{A}_j - \mathcal{C}_j, i.o.) = \mathbf{P}(\mathcal{B}_j - \mathcal{C}_j, i.o.) = 0.$$

We will now show that $\mathbf{P}(\mathcal{C}_j, i.o.) = 0$. Note that since $|T_{j,r_j}^{(\rho_1)}| \leq |T_{j,d_j}^{(\rho_1)}|$ and

$$T_{q^j} = Y_{q^j}(\rho_1 - \hat{\rho}_{1,q^{j-1}}) + Y_{q^j}(\hat{\rho}_{1,q^{j-1}} - Z_{q^j}) = T_{j-1,d_{j-1}}^{(\rho_1)} + \tilde{T}_{j-1,d_{j-1}},$$

it follows that

$$|T_{j,r_j}^{(\rho_1)} - T_{q^j}| \leq |T_{j,d_j}^{(\rho_1)}| + |T_{j-1,d_{j-1}}^{(\rho_1)}| + |\tilde{T}_{j-1,d_{j-1}}|,$$

which implies that

$$\{\mathcal{C}_j, i.o.\} \subset \left\{ |T_{j,d_j}^{(\rho_1)}| > \frac{\epsilon}{3} j^2 q^{\frac{j}{2}}, i.o. \right\} \cup \left\{ |\tilde{T}_{j,d_j}| > \frac{\epsilon}{3} j^2 q^{\frac{j}{2}}, i.o. \right\}.$$

Now, since $Y_n \leq Y_0 + bn$, it follows that

$$\{\mathcal{C}_j, i.o.\} \subset \{\mathcal{G}_{1j}, i.o.\} \cup \{\mathcal{G}_{2j}, i.o.\},$$

where

$$\begin{aligned} \mathcal{G}_{1j} &:= \left\{ \left(\frac{Y_0}{bq^{j+1}} + 1 \right) q^{\frac{j}{2}} |\rho_1 - \hat{\rho}_{1,q^j}| > j^2 \frac{\epsilon}{3qb} \right\}, \\ \mathcal{G}_{2j} &:= \left\{ \left(\frac{Y_0}{bq^{j+1}} + 1 \right) q^{\frac{j}{2}} |Z_{q^{j+1}} - \hat{\rho}_{1,q^j}| > j^2 \frac{\epsilon}{3bq} \right\}. \end{aligned}$$

We will now show that $\mathbf{P}(\mathcal{G}_{1j}, i.o.) = 0$. By using the Markov's inequality we have

$$\begin{aligned} \sum_{j=1}^{\infty} \mathbf{P}(\mathcal{G}_{1j}) &\leq \frac{3qb}{\epsilon} \sum_{j=1}^{\infty} \left(\frac{Y_0}{bq^{j+1}} + 1 \right) \frac{\mathbf{E} \left[q^{\frac{j}{2}} |\rho_1 - \hat{\rho}_{1,q^j}| \right]}{j^2} \\ &= \frac{3qb}{\epsilon} \left(\frac{Y_0}{bq} + 1 \right) C \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty, \end{aligned}$$

where

$$C := \sup_{k \geq 1} \mathbf{E} \left\{ \left[q^{\frac{k}{2}} |\rho_1 - \hat{\rho}_{1,q^k}| \right] \right\} < \infty$$

from (2.7). Hence, using the Borel-Cantelli lemma, it follows that $\mathbf{P}(\mathcal{G}_{1j}, i.o.) = 0$.

Now, consider \mathcal{G}_{2j} . Let $\mathcal{H}_j := \{j^{-2} q^{\frac{j}{2}} \cdot |\Delta_{j,d_j}| > \epsilon\}$ and since

$$\mathbf{P}(\mathcal{G}_{2j}, i.o.) = \mathbf{P}(\mathcal{H}_j, i.o.)$$

we now focus on \mathcal{H}_j . First, for each $j \geq 1$, we recall that $\mathcal{Q}_j = \{\tau_j > d_j\}$ and we decompose \mathcal{H}_j as follows:

$$\mathcal{H}_j \subseteq \mathcal{Q}_j \cup \{\mathcal{H}_j \cap \mathcal{Q}_j^C\},$$

which leads to

$$\mathbf{P}(\mathcal{H}_j, i.o.) \leq \mathbf{P}(\mathcal{Q}_j, i.o.) + \mathbf{P}(\mathcal{H}_j \cap \mathcal{Q}_j^C, i.o.).$$

First, consider $\mathbf{P}(\mathcal{H}_j \cap \mathcal{Q}_j^C, i.o.)$. By using Markov's inequality we have

$$\begin{aligned} \sum_{j=1}^{\infty} \mathbf{P}(\mathcal{H}_j \cap \mathcal{Q}_j^C) &\leq \sum_{j=1}^{\infty} \mathbf{E} \left[q^j \cdot |\Delta_{j,d_j}| \mathbf{1}_{\mathcal{Q}_j^C} \right] \frac{q^{-\frac{j}{2}}}{\epsilon j^2} \\ &\leq \frac{\left(\sup_{k \geq 1} \left\{ \mathbf{E} \left[q^k \cdot |\Delta_{k,d_k}| \mathbf{1}_{\mathcal{Q}_k^C} \right] \right\} \right)}{\epsilon} \sum_{j=1}^{\infty} \frac{q^{-\frac{j}{2}}}{j^2}, \end{aligned}$$

which is finite from Theorem 3.2. Hence, again from the Borel-Cantelli lemma we have that

$$\mathbf{P}(\mathcal{H}_j \cap \mathcal{Q}_j^C, i.o.) = 0.$$

We will now show that $\mathbf{P}(\mathcal{Q}_j, i.o.) = 0$. To this end, we can follow the same arguments used in the first part of this proof, except that here we define

$$\mathcal{C}_j := \left\{ |T_{j,d_j}^{(\rho_1)} - T_{q^j}| > \epsilon q^j \right\}.$$

In this case, to show $\mathbf{P}(\mathcal{C}_j, i.o.) = 0$ we have to prove that the following two events cannot occur infinitely often

- (i) $\mathcal{G}_{3j} := \left\{ \left(\frac{Y_0}{bq^{j+1}} + 1 \right) |\rho_1 - \hat{\rho}_{1,q^j}| > \frac{\epsilon}{2qb} \right\},$
- (ii) $\mathcal{G}_{4j} := \left\{ \left(\frac{Y_0}{bq^j} + 1 \right) |\rho_1 - Z_{q^j}| > \frac{\epsilon}{2b} \right\}.$

Result (i) is implied by (2.7), while (ii) follows from Theorem 2.1. Hence, we have that

$$\mathbf{P}(\mathcal{C}_j, i.o.) = 0.$$

Then, similarly to the first part of the proof, we deal with the sets $\mathcal{A}_j - \mathcal{C}_j$ and $\mathcal{B}_j - \mathcal{C}_j$ by applying Chernoff's upper bound to the probabilities

$$P_{A_j} = \mathbf{P} \left(\sum_{i=1}^{d_j} A_i < \epsilon q^j \right) \quad \text{and} \quad P_{B_j} = \mathbf{P} \left(\sum_{i=1}^{d_j} B_i > -\epsilon q^j \right),$$

which implies $\sum_{j=1}^{\infty} P_{A_j} < \infty$ and $\sum_{j=1}^{\infty} P_{B_j} < \infty$. Hence, from the Borel-Cantelli lemma we get

$$\mathbf{P}(\mathcal{A}_j - \mathcal{C}_j, i.o.) = \mathbf{P}(\mathcal{B}_j - \mathcal{C}_j) = 0.$$

which implies $\mathbf{P}(\mathcal{Q}_j, i.o.) = 0$. This concludes the proof. \square

REMARK 4.1. *The result of Theorem 4.1 continues to hold if (2.7) is not satisfied, but (2.2) and condition (c1) hold. Moreover, since in the proof we use Theorem 3.1, if (2.7) does not hold condition (c2) must be assumed (see Remark 3.1).*

4.3. Proof of Proposition 2.1.

PROOF. Wlog assume $m_1 > m_2$, which implies $\hat{\rho}_n = \hat{\rho}_{1,n}$ and $\rho = \rho_1$. First, we have

$$(4.15) \quad \mathbf{E} [n|\bar{\rho}_{1,n} - \rho_1|^2] = \frac{1}{n} \mathbf{E} \left[\left| \sum_{i=0}^{n-1} (\tilde{\rho}_{1,i} - \rho_1) \right|^2 \right],$$

and note that

$$\begin{aligned} \sum_{i=0}^{n-1} (\tilde{\rho}_{1,i} - \rho_1) &= \sum_{j=0}^{k_n} \sum_{i=0}^{d_j} (\tilde{\rho}_{1,q^j+i} - \rho_1) \mathbf{1}_{\{q^{k_n}+i \leq n\}} \\ &= \sum_{j=0}^{k_n-1} d_j (\hat{\rho}_{1,q^j} - \rho_1) + (n - q^{k_n}) (\hat{\rho}_{1,q^{k_n}} - \rho_1), \end{aligned}$$

where we recall k_n is defined in (2.6) as $k_n := \lfloor \log_q(n) \rfloor$. Since $d_j = (q-1)q^j$, the LHS of (4.15) is equal to

$$\frac{(q-1)^2}{n} \mathbf{E} \left[\left| \sum_{j=0}^{k_n-1} (\sqrt{q})^j \cdot \left(\sqrt{q}^j (\hat{\rho}_{1,q^j} - \rho_1) \right) + \left(\frac{n - q^{k_n}}{q-1} \right) (\hat{\rho}_{1,q^{k_n}} - \rho_1) \right|^2 \right],$$

and, defining $c_j := \sqrt{q}^j |\hat{\rho}_{1,q^j} - \rho_1|$, we can rewrite the last expression as follows:

$$\frac{(q-1)^2}{n} \mathbf{E} \left[\left(\sum_{j=0}^{k_n-1} (\sqrt{q})^j \cdot c_j + \left[\frac{n - q^{k_n}}{\sqrt{q}^{k_n} (q-1)} \right] c_{k_n} \right)^2 \right];$$

Now, using Cauchy Schwartz inequality and using $\left(\frac{n - q^{k_n}}{\sqrt{q}^{k_n} (q-1)} \right) \leq \sqrt{q}^{k_n}$, the above expectation is less than or equal to

$$K_n := \sum_{j_1=0}^{k_n} \sum_{j_2=0}^{k_n} (\sqrt{q})^{j_1} (\sqrt{q})^{j_2} \cdot \sqrt{\mathbf{E} [c_{j_1}^2] \mathbf{E} [c_{j_2}^2]}.$$

Now, by the symmetry in K_n , we can use the following decomposition

$$\begin{aligned} \sum_{j_1=0}^{k_n} \sum_{j_2=0}^{k_n} (\cdot) &= \sum_{j_1=0}^{\sqrt{k_n}} \sum_{j_2=0}^{\sqrt{k_n}} (\cdot) + 2 \sum_{j_1=0}^{\sqrt{k_n}} \sum_{j_2=\sqrt{k_n}}^{k_n} (\cdot) + \sum_{j_1=\sqrt{k_n}}^{k_n} \sum_{j_2=\sqrt{k_n}}^{k_n} (\cdot) \\ &\leq 2 \sum_{j_1=0}^{\sqrt{k_n}} \sum_{j_2=0}^{\sqrt{k_n}} (\cdot) + \sum_{j_1=\sqrt{k_n}}^{k_n} \sum_{j_2=\sqrt{k_n}}^{k_n} (\cdot), \end{aligned}$$

we obtain

$$\begin{aligned}
K_n &\leq \sup_{j \geq 1} \{ \mathbf{E} [c_j^2] \} \cdot 2 \sum_{j_1=0}^{k_n} \sum_{j_2=0}^{\sqrt{k_n}} (\sqrt{q})^{j_1} (\sqrt{q})^{j_2} \\
&\quad + \max_{\sqrt{k_n} \leq j \leq k_n} \{ \mathbf{E} [c_j^2] \} \cdot \sum_{j_1=\sqrt{k_n}}^{k_n} \sum_{j_2=\sqrt{k_n}}^{k_n} (\sqrt{q})^{j_1} (\sqrt{q})^{j_2} \\
&= K_{1n} + K_{2n}.
\end{aligned}$$

Now, consider K_{1n} ; we have that

$$K_{1n} \leq \sup_{j \geq 1} \{ \mathbf{E} [c_j^2] \} \cdot 2 \left(\frac{(\sqrt{q})^{\sqrt{k_n}+1} - 1}{\sqrt{q} - 1} \right) \left(\frac{(\sqrt{q})^{k_n+1} - 1}{\sqrt{q} - 1} \right),$$

and by multiplying for $(q-1)^2/n$ we obtain

$$2 \left(\frac{q-1}{\sqrt{q}-1} \right)^2 \sup_{j \geq 1} \{ \mathbf{E} [c_j^2] \} \cdot \left(\frac{(\sqrt{q})^{\sqrt{k_n}+1} - 1}{\sqrt{n}} \right) \left(\frac{(\sqrt{q})^{k_n+1} - 1}{\sqrt{n}} \right).$$

Using (2.7) we have that $\sup_{j \geq 1} \{ \mathbf{E} [c_j^2] \}$ is finite. Moreover, since $n \leq q^{k_n+1}$ by definition of k_n , we have that

$$\left(\frac{(\sqrt{q})^{k_n+1} - 1}{\sqrt{n}} \right) \leq \sqrt{q}, \quad \left(\frac{(\sqrt{q})^{\sqrt{k_n}+1} - 1}{\sqrt{n}} \right) \rightarrow 0.$$

Similarly, we can consider K_{2n} and write

$$K_{2n} \leq \max_{\sqrt{k_n} \leq j \leq k_n} \{ \mathbf{E} [c_j^2] \} \cdot \left(\frac{(\sqrt{q})^{k_n+1} - 1}{\sqrt{q} - 1} \right)^2.$$

Then, by multiplying for $(q-1)^2/n$ we obtain

$$\left(\frac{q-1}{\sqrt{q}-1} \right)^2 \max_{\sqrt{k_n} \leq j \leq k_n} \{ \mathbf{E} [c_j^2] \} \cdot \left(\frac{(\sqrt{q})^{k_n+1} - 1}{\sqrt{n}} \right)^2,$$

and from (2.7) and $n \leq q^{k_n+1}$ we have $\max_{\sqrt{k_n} \leq j \leq k_n} \{ \mathbf{E} [c_j^2] \}$ is finite and

$$\left(\frac{(\sqrt{q})^{k_n+1} - 1}{\sqrt{n}} \right)^2 \leq q.$$

Then, combining all together we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbf{E} [n|\bar{\rho}_{1,n} - \rho_1|^2] &\leq \limsup_{n \rightarrow \infty} \frac{(q-1)^2}{n} K_{2n} \\ &\leq q(1 + \sqrt{q})^2 \cdot \limsup_{n \rightarrow \infty} \mathbf{E} [n|\hat{\rho}_{1,n} - \rho_1|^2], \end{aligned}$$

which is finite because of condition (2.7). \square

4.4. Proof of Corollary 2.2.

PROOF. To prove this result, we apply Theorem 2.2 to the urn model with fixed thresholds, i.e. $\tilde{\rho}_{1,n} = \rho_1$ and $\tilde{\rho}_{2,n} = \rho_2$ for all $n \geq 0$, since in this case $\bar{\rho}_n = \rho$ for all $n \geq 0$. \square

4.5. *Remarks on the CLT for Z_n .* In this subsection, we discuss the second-order behavior of the proportion Z_n of balls in the urn in the ARRU model. Specifically, we establish CLT of Z_{n_j} for some specific subsequences $\{n_j; j \geq 1\}$ and we highlight the challenges to the proof of a full CLT. To this end, let us assume that

$$(4.16) \quad \sqrt{n}(\hat{\rho}_n - \rho) \xrightarrow{d} Z,$$

where Z is a Gaussian random variable with zero mean and variance $\sigma^2 > 0$. It is worth noticing that (4.16) is usually verified in applications, since $\hat{\rho}_n$ is typically a continuous function of maximum likelihood estimators, e.g. see (2.10).

Wlog, assume $m_1 > m_2$ and consider the sequence $\{\sqrt{n}(\rho_1 - Z_n); n \geq 1\}$.

Now, along the subsequence $\{q^j; j \geq 1\}$

$$(4.17) \quad \sqrt{q^j}(\rho_1 - Z_{q^j}) = \sqrt{q^j}(\rho_1 - \hat{\rho}_{1,q^{j-1}}) + \sqrt{q^j}(\hat{\rho}_{1,q^{j-1}} - Z_{q^j}).$$

Using (4.16), it follows that

$$\sqrt{q^j}(\rho_1 - \hat{\rho}_{1,q^{j-1}}) = \sqrt{q} \cdot \left[\sqrt{q^{j-1}}(\rho_1 - \hat{\rho}_{1,q^{j-1}}) \right] \xrightarrow{d} \sqrt{q} \cdot Z.$$

As for the second term in (4.17), it can be expressed as

$$\begin{aligned} \sqrt{q^j}(\hat{\rho}_{1,q^{j-1}} - Z_{q^j}) &= \sqrt{q^j} \Delta_{j-1, d_{j-1}} \\ &= \sqrt{q^j} \Delta_{j-1, d_{j-1}} \mathbf{1}_{\mathcal{R}_{j-1}} + \sqrt{q^j} \Delta_{j-1, d_{j-1}} \mathbf{1}_{\mathcal{R}_{j-1}^c}, \end{aligned}$$

where we recall $\mathcal{R}_j := \{\tau_j > r_j\}$ and $r_j := q^{j(1+\nu)/2}$, with $\nu \in (0, 1/2)$. The first term converges to zero a.s. from Theorem 4.1, while using $r_{j-1} \leq d_{j-1}$ from Theorem 3.2 we have

$$\sqrt{q^j} \mathbf{E} \left[|\Delta_{j-1, d_{j-1}}| \mathbf{1}_{\mathcal{R}_{j-1}^c} \right] \leq \sqrt{q^j} \mathbf{E} \left[|\Delta_{j-1, d_{j-1}}| \mathbf{1}_{\{\tau_{j-1} \leq d_{j-1}\}} \right] \rightarrow 0.$$

Thus,

$$\sqrt{q^j}(\rho_1 - Z_{q^j}) \xrightarrow{d} \sqrt{q} \cdot Z.$$

Now, consider the subsequence $\{q^j + r_j; j \geq 1\}$. Now,

$$(4.18) \quad \sqrt{q^j}(\rho_1 - Z_{q^j+r_j}) = \sqrt{q^j}(\rho_1 - \hat{\rho}_{1, q^j}) + \sqrt{q^j}(\hat{\rho}_{1, q^j} - Z_{q^j+r_j}).$$

As before, from (4.16) we have that

$$\sqrt{q^j}(\rho_1 - \hat{\rho}_{1, q^j}) \xrightarrow{d} Z.$$

Once again expressing the second term in (4.17) below,

$$\begin{aligned} \sqrt{q^j}(\hat{\rho}_{1, q^j} - Z_{q^j+r_j}) &= \sqrt{q^j} \Delta_{j, r_j} \\ &= \sqrt{q^j} \Delta_{j, r_j} \mathbf{1}_{\mathcal{R}_j} + \sqrt{q^j} \Delta_{j, r_j} \mathbf{1}_{\mathcal{R}_j^c}, \end{aligned}$$

one can show the first term in the RHS converges to zero a.s. from Theorem 4.1. The second term in the RHS tends to zero in L^1 from Theorem 3.2. Thus,

$$\sqrt{q^j + r_j}(\rho_1 - Z_{q^j+r_j}) \xrightarrow{d} Z.$$

A crucial result to obtain CLT for $\{Z_{n_j}; j \geq 1\}$, with $n_j = q^j$ and $n_j = q^j + r_j$, is Theorem 4.1, which establishes $\mathbf{P}(\tau_j > n_j, i.o.) = 0$. From these results, it follows that the asymptotic distribution of Z_{n_j} only involves times $t \in \cup_j (q^j + \tau_j, q^{j+1})$; for these times, Theorem 3.2 establishes a uniform bound for $n|Z_n - \tilde{\rho}_n|$. However, at times $t \in \cup_j (q^j, q^j + \tau_j)$ it seems difficult to obtain the detailed behavior of $(Z_n - \tilde{\rho}_n)$. This gap needs to be handled for a CLT for $\{Z_n; n \geq 1\}$. This is beyond the scope of the current paper.

5. Simulation studies. In this section, we describe some simulation studies that illustrate the theoretical results presented in Section 2 in the context of clinical trials. We recall from subsection 2.3 that, in the context of clinical trials, the random variables $\xi_{1,n}$ and $\xi_{2,n}$ are interpreted as potential responses to competing treatments \mathcal{T}_1 and \mathcal{T}_2 , whose distributions μ_1 and μ_2 depend on parameters $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ respectively. Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$. Now, letting f_1 and f_2 are two continuous functions, we recall that $\rho_1 = f_1(\boldsymbol{\theta})$

and $\rho_2 = f_2(\boldsymbol{\theta})$. Moreover, the adaptive thresholds $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ are defined as follows:

$$\hat{\rho}_{1,n} := f_1(\hat{\boldsymbol{\theta}}_{1,n}) \quad \text{and} \quad \hat{\rho}_{2,n} := f_2(\hat{\boldsymbol{\theta}}_{2,n}), \quad \forall n \geq 1,$$

where $\hat{\boldsymbol{\theta}}_{1,n}$ and $\hat{\boldsymbol{\theta}}_{2,n}$ are the adaptive estimators of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ after the first n allocations.

The main goal of this section is to illustrate the asymptotic behavior of the allocation proportion $N_{1,n}/n$ and of the parameter estimators $\hat{\boldsymbol{\theta}}_n$. Simulations are performed with $N = 10^5$ independent urn processes, each which evolve following the model described in Section 2 with adaptive thresholds $\tilde{\rho}_{1,n}$ and $\tilde{\rho}_{2,n}$ that change at exponential times $\{q^j; j \geq 1\}$, with $q = 1.25$, [see (2.5)]. For all the N urn processes we used initial composition $(y_{1,0}, y_{2,0}) = (2, 2)$ and sample size $n = 200$. The functions f_1 and f_2 are chosen as in (2.11) with $p = 0.75$. We analyze both Bernoulli and Gaussian responses.

5.1. Bernoulli responses. We assume responses to treatments \mathcal{T}_1 and \mathcal{T}_2 are from Bernoulli distributions with parameters p_1 and p_2 , respectively. In this case, $\boldsymbol{\theta} = (p_1, p_2)$ is the vector of unknown parameters. We examine two target allocations:

- (a) $\eta(\boldsymbol{\theta}) = (1 - p_1) / (2 - p_1 - p_2)$, proposed by [22];
- (b) $\eta(\boldsymbol{\theta}) = \sqrt{p_1} / (\sqrt{p_1} + \sqrt{p_2})$, proposed by [20].

Hence, from (2.11) with $p = 0.75$, we have

$$\rho_1 = 0.25 \cdot 1 + 0.75 \cdot \eta(p_1, p_2), \quad \text{and} \quad \rho_2 = 0.25 \cdot 0 + 0.75 \cdot \eta(p_1, p_2).$$

In Table 5.1 we report the simulation results on the mean and standard error of the allocation proportion $N_{1,n}/n$ and of the estimators $\hat{p}_{1,n}$ and $\hat{p}_{2,n}$, defined as

$$\hat{p}_{1,n} = \frac{\sum_{i=1}^n X_i \xi_{1,i}}{N_{1,n}}, \quad \text{and} \quad \hat{p}_{2,n} = \frac{\sum_{i=1}^n (1 - X_i) \xi_{2,i}}{N_{2,n}}.$$

Hence,

$$\hat{\rho}_{1,n} = 0.25 \cdot 1 + 0.75 \cdot \eta(\hat{p}_{1,n}, \hat{p}_{2,n}),$$

and

$$\hat{\rho}_{2,n} = 0.25 \cdot 0 + 0.75 \cdot \eta(\hat{p}_{1,n}, \hat{p}_{2,n}).$$

TABLE 1

Simulation of $N_{1,n}/n$ and $\hat{\theta}_n$ are given for different designs, with mean square errors given in parenthesis. The target allocation is $\rho_1 = (1 - p) \cdot 1 + p \cdot \eta(p_1, p_2)$ with $p = 0.75$. Simulation used $N = 10^5$ ARRU processes with $n = 200$ and changes at times $\{q^j; j \geq 1\}$ with $q = 1.25$. Initial composition $(y_{1,0}, y_{2,0}) = (2, 2)$.

p_1	p_2	ρ_1	$N_{1,n}/n$	$\hat{p}_{1,n}$	$\hat{p}_{2,n}$
(a) $\eta = (1 - p_1)/(2 - p_1 - p_2)$					
0.9	0.7	0.44	0.44(0.07)	0.89(0.03)	0.7(0.04)
0.9	0.5	0.38	0.41(0.06)	0.89(0.03)	0.50(0.05)
0.9	0.3	0.34	0.40(0.07)	0.89(0.03)	0.30(0.04)
0.9	0.1	0.33	0.43(0.12)	0.89(0.03)	0.11(0.03)
0.7	0.5	0.53	0.50(0.07)	0.70(0.05)	0.50(0.05)
0.7	0.3	0.48	0.48(0.05)	0.70(0.05)	0.30(0.04)
0.7	0.1	0.44	0.48(0.06)	0.70(0.05)	0.11(0.03)
0.5	0.3	0.56	0.53(0.06)	0.50(0.05)	0.30(0.05)
0.5	0.1	0.52	0.53(0.04)	0.50(0.05)	0.11(0.03)
0.3	0.1	0.58	0.56(0.05)	0.30(0.04)	0.11(0.03)
(b) $\eta = \sqrt{p_1}/(\sqrt{p_1} + \sqrt{p_2})$					
0.9	0.7	0.65	0.57(0.11)	0.89(0.03)	0.69(0.05)
0.9	0.5	0.68	0.63(0.08)	0.89(0.03)	0.50(0.06)
0.9	0.3	0.73	0.69(0.06)	0.89(0.03)	0.30(0.06)
0.9	0.1	0.81	0.76(0.07)	0.89(0.02)	0.11(0.04)
0.7	0.5	0.66	0.58(0.11)	0.69(0.04)	0.50(0.06)
0.7	0.3	0.70	0.66(0.07)	0.70(0.04)	0.30(0.06)
0.7	0.1	0.79	0.74(0.07)	0.70(0.04)	0.12(0.04)
0.5	0.3	0.67	0.60(0.10)	0.50(0.05)	0.30(0.05)
0.5	0.1	0.77	0.70(0.08)	0.50(0.04)	0.11(0.04)
0.3	0.1	0.73	0.64(0.11)	0.30(0.04)	0.11(0.03)

5.2. *Gaussian responses.* We now assume responses to treatments \mathcal{T}_1 and \mathcal{T}_2 are from a Gaussian distribution with parameters (m_1, σ_1^2) and (m_2, σ_2^2) , respectively. In this case, $\theta = (m_1, \sigma_1^2, m_2, \sigma_2^2)$ is the vector of unknown parameters. We examine two target allocation:

- (c) $\eta(\theta) = \sigma_1/(\sigma_1 + \sigma_2)$, used in [14];
- (d) $\eta(\theta) = \sigma_1\sqrt{m_2}/(\sigma_1\sqrt{m_2} + \sigma_2\sqrt{m_1})$, proposed by [23].

Hence, from (2.11) with $p = 0.75$, we have

$$\rho_1 = 0.25 \cdot 1 + 0.75 \cdot \eta(\theta), \quad \text{and} \quad \rho_2 = 0.25 \cdot 0 + 0.75 \cdot \eta(\theta).$$

In Table 5.2 we report the simulation results on the mean and standard error of the allocation proportion $N_{1,n}/n$ and the parameter estimators $\hat{\sigma}_{1,n}^2$ and

TABLE 2

Simulations of $N_{1,n}/n$ and $\hat{\theta}_n$ are given for different designs, with mean square errors given in parenthesis. The target allocation is $\rho_1 = (1 - p) \cdot 1 + p \cdot \eta(\theta)$ with $p = 0.75$. Simulation used $N = 10^5$ ARRU processes with $n = 200$ and changes at times $\{q^j; j \geq 1\}$ with $q = 1.25$. Initial composition $(y_{1,0}, y_{2,0}) = (2, 2)$.

m_1	m_2	σ_1^2	σ_2^2	ρ_1	$N_{1,n}/n$	$\hat{\sigma}_{1,n}^2$	$\hat{\sigma}_{2,n}^2$
(c) $\eta = \sigma_1 / (\sigma_1 + \sigma_2)$							
10	5	1	1	0.63	0.61(0.05)	1.01(0.13)	1.01(0.16)
8	5	1	1	0.63	0.59(0.07)	1.01(0.13)	1.01(0.16)
6	5	1	1	0.63	0.55(0.12)	1.01(0.14)	1.01(0.15)
10	5	4	1	0.75	0.73(0.06)	4.00(0.47)	1.01(0.20)
8	5	4	1	0.75	0.71(0.07)	4.00(0.48)	1.01(0.19)
6	5	4	1	0.75	0.66(0.13)	4.03(0.50)	1.01(0.18)
10	5	1	4	0.50	0.49(0.05)	1.01(0.14)	4.00(0.57)
8	5	1	4	0.50	0.48(0.07)	1.01(0.15)	4.03(0.56)
6	5	1	4	0.50	0.43(0.11)	1.01(0.16)	4.03(0.54)
(d) $\eta = \sigma_1 \sqrt{m_2} / (\sigma_1 \sqrt{m_2} + \sigma_2 \sqrt{m_1})$							
10	5	1	1	0.56	0.55(0.05)	1.01(0.14)	1.01(0.15)
8	5	1	1	0.58	0.55(0.07)	1.01(0.14)	1.01(0.15)
6	5	1	1	0.61	0.53(0.12)	1.01(0.14)	1.01(0.15)
10	5	4	1	0.69	0.67(0.06)	4.03(0.49)	1.01(0.18)
8	5	4	1	0.71	0.67(0.07)	4.03(0.49)	1.01(0.18)
6	5	4	1	0.73	0.65(0.13)	4.03(0.51)	1.01(0.18)
10	5	1	4	0.45	0.44(0.05)	1.01(0.15)	4.03(0.54)
8	5	1	4	0.46	0.44(0.07)	1.01(0.16)	4.03(0.54)
6	5	1	4	0.48	0.42(0.11)	1.01(0.16)	4.03(0.53)

$\hat{\sigma}_{2,n}^2$, defined as

$$\hat{\sigma}_{1,n}^2 = \frac{\sum_{i=1}^n X_i (\xi_{1,i} - \hat{m}_{1,n})^2}{N_{1,n}} \quad \text{and}$$

$$\hat{\sigma}_{2,n}^2 = \frac{\sum_{i=1}^n (1 - X_i) (\xi_{2,i} - \hat{m}_{2,n})^2}{N_{2,n}},$$

where $\hat{m}_{1,n} = \sum_{i=1}^n X_i \xi_{1,i} / N_{1,n}$ and $\hat{m}_{2,n} = \sum_{i=1}^n (1 - X_i) \xi_{2,i} / N_{2,n}$. Hence,

$$\hat{\rho}_{1,n} = 0.25 \cdot 1 + 0.75 \cdot \eta(\hat{\theta}_n),$$

and

$$\hat{\rho}_{2,n} = 0.25 \cdot 0 + 0.75 \cdot \eta(\hat{\theta}_n).$$

The results show that our methods target the true parameters effectively. In real clinical trials, further calibration may be performed to reduce small bias.

6. Extensions to multi-color urn models. It is important to note that all the results presented in this paper can be extended to the case of $K > 2$ colors, when $\exists j \in \{1, \dots, K\}$ such that $m_j > m_k$ for any $k \neq j$. In the context of clinical trials, the functions f_j should be interpreted as the target allocations for $N_{j,n}/n$ when \mathcal{T}_j is the superior treatment, and the variables $W_{j,n}$ should be all defined as $\mathbf{1}_{\{Z_n \leq \hat{\rho}_{j,n}\}}$.

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